CAN 1011: Data Communication

- Fourier Series Analysis
- Fourier Transform Representation

Contents

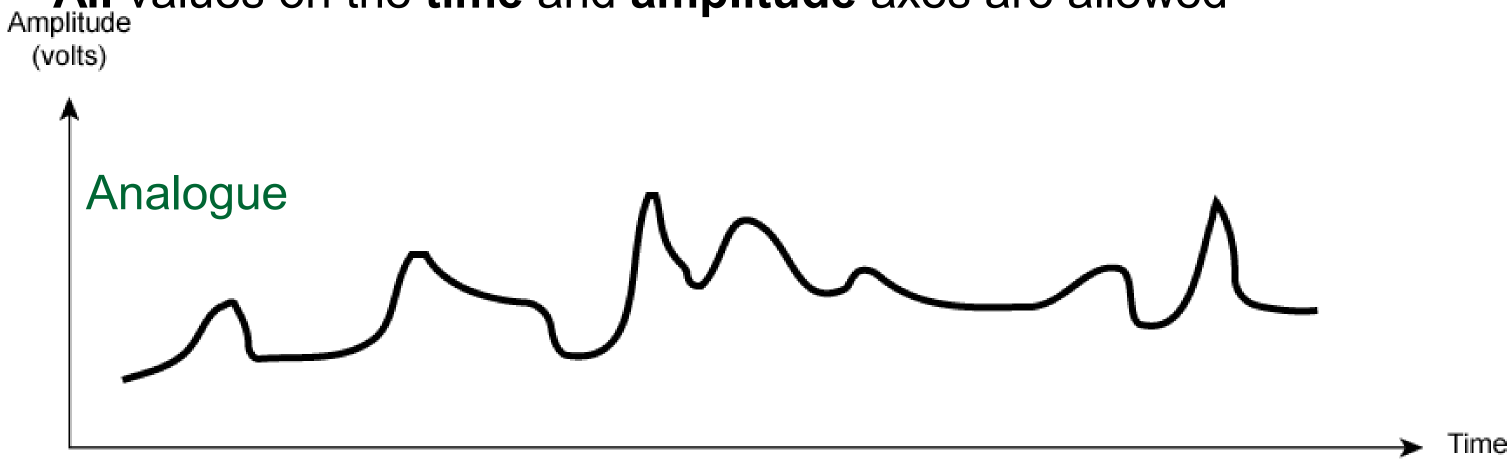
- Concepts & Terminology
- Signal representation: Time and Frequency domains
- Fourier Series
- Fourier Transformation

Frequency, Spectrum & Bandwidth

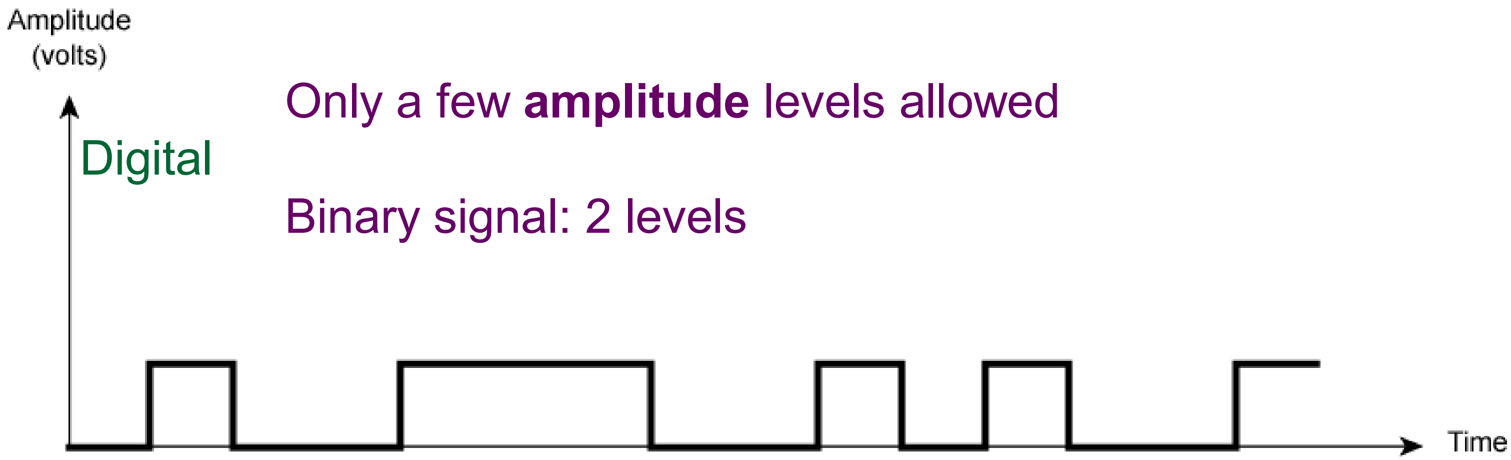
- Time-domain concepts
  - Analogue signal
    - Varies in a smooth, continuous way in both time and amplitude
  - Digital signal
    - Maintains a constant level for some time and then changes to another constant level (i.e. amplitude takes only a finite number of discrete levels)
  - Periodic signal
    - Same pattern repeated over time, e.g. sine wave or a square wave
  - Aperiodic signal
    - Pattern not repeated over time

Analogue & Digital Signals

All values on the **time** and **amplitude** axes are allowed



(a) Analog



(b) Digital

Periodic Signals

For any periodic wave:

$S(t+nT) = S(t); \quad 0 \le t \le T$

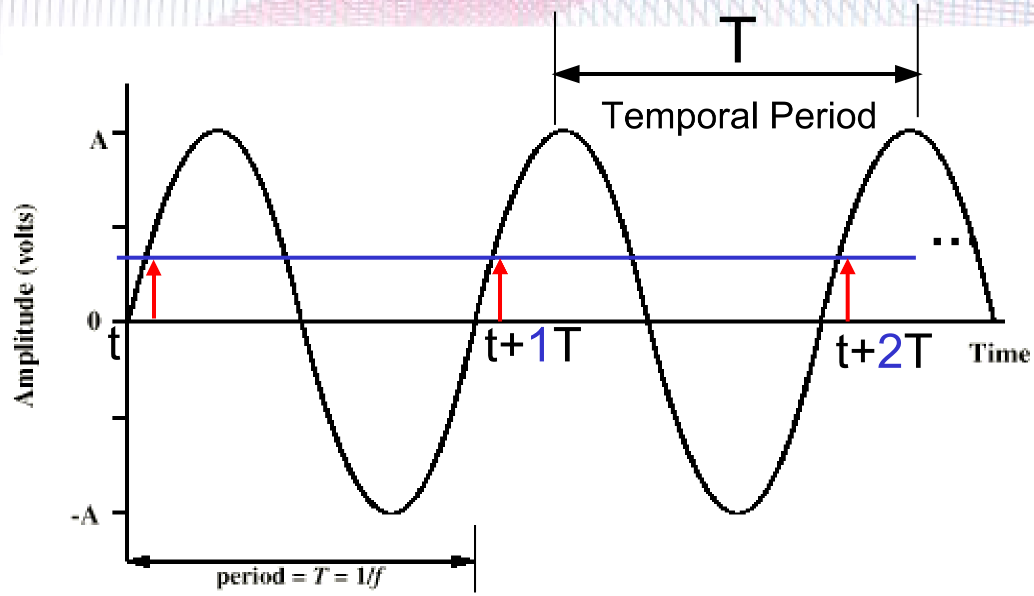
where

t is time over first period

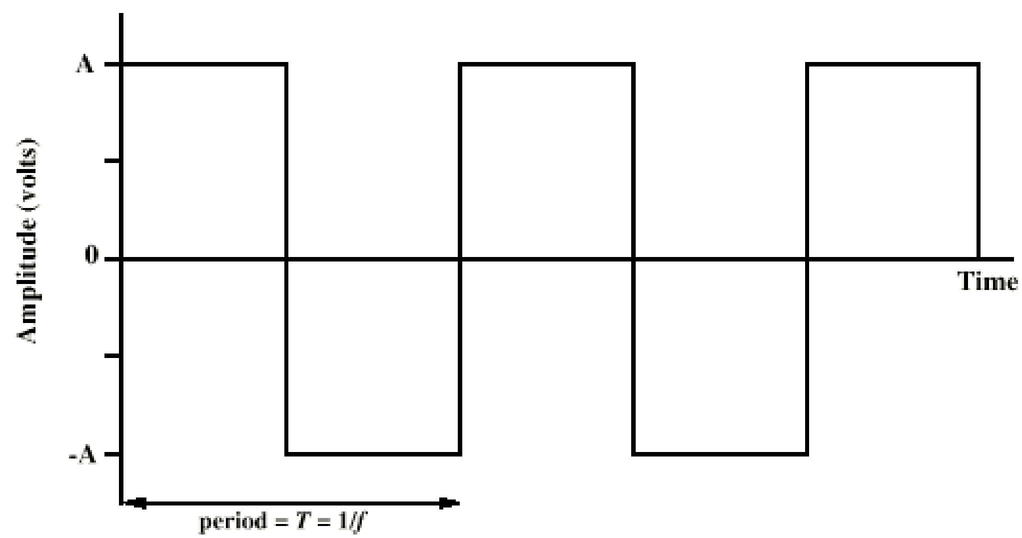
T is the waveform period

n is an integer

Signal behaviour over **one period** describes behaviour at all times

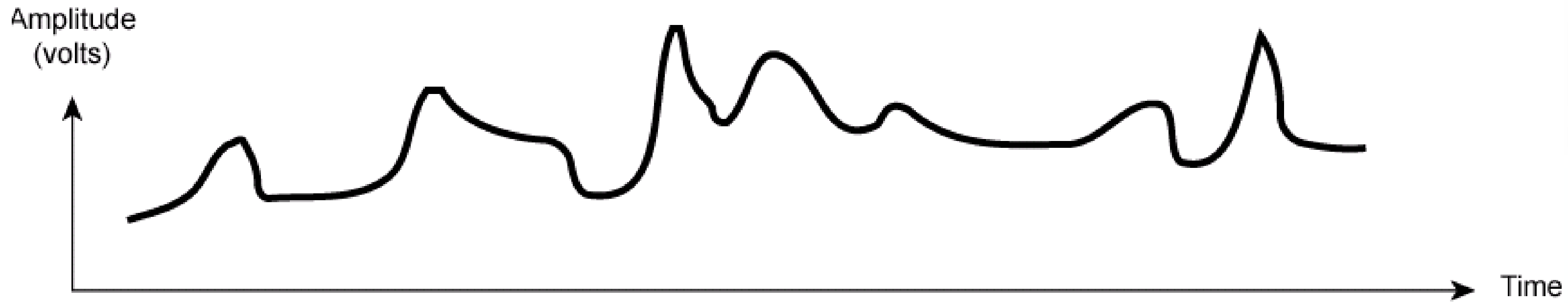


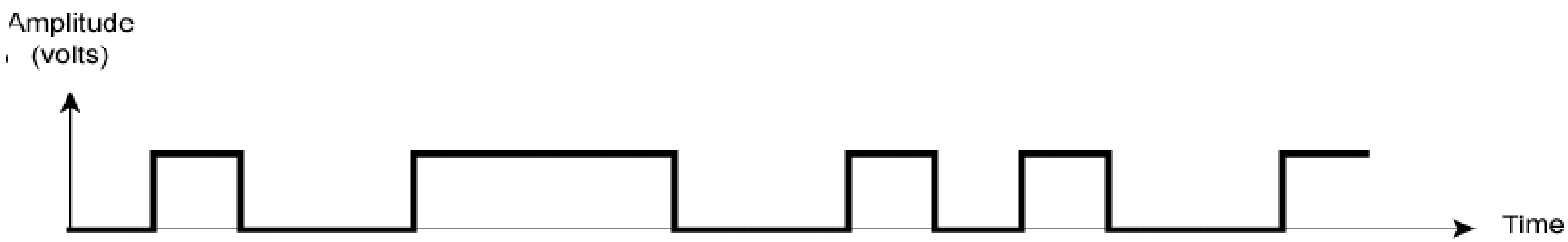
(a) Sine wave

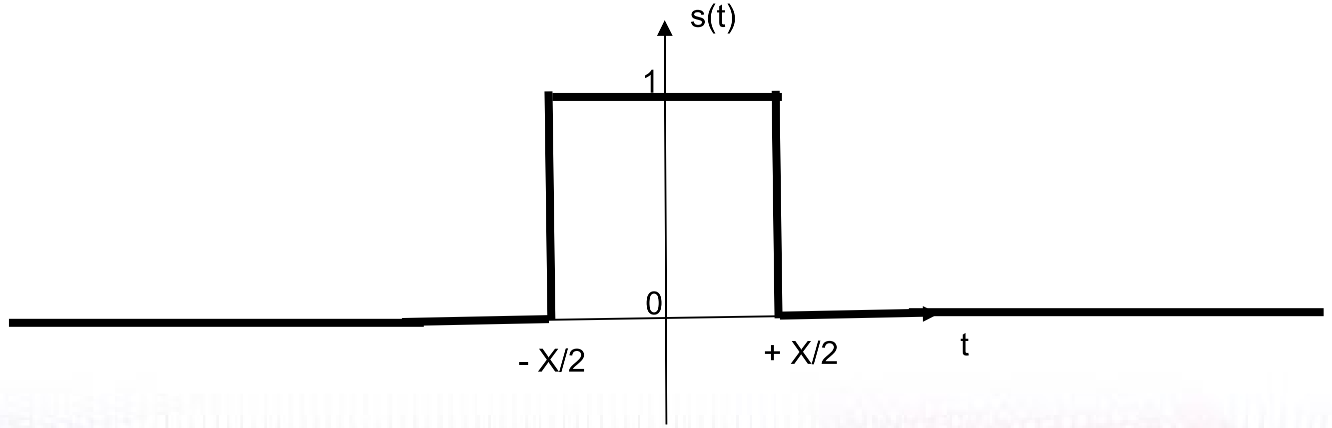


(b) Square wave

Aperiodic (non periodic) signals in time







Continuous Versus Discrete

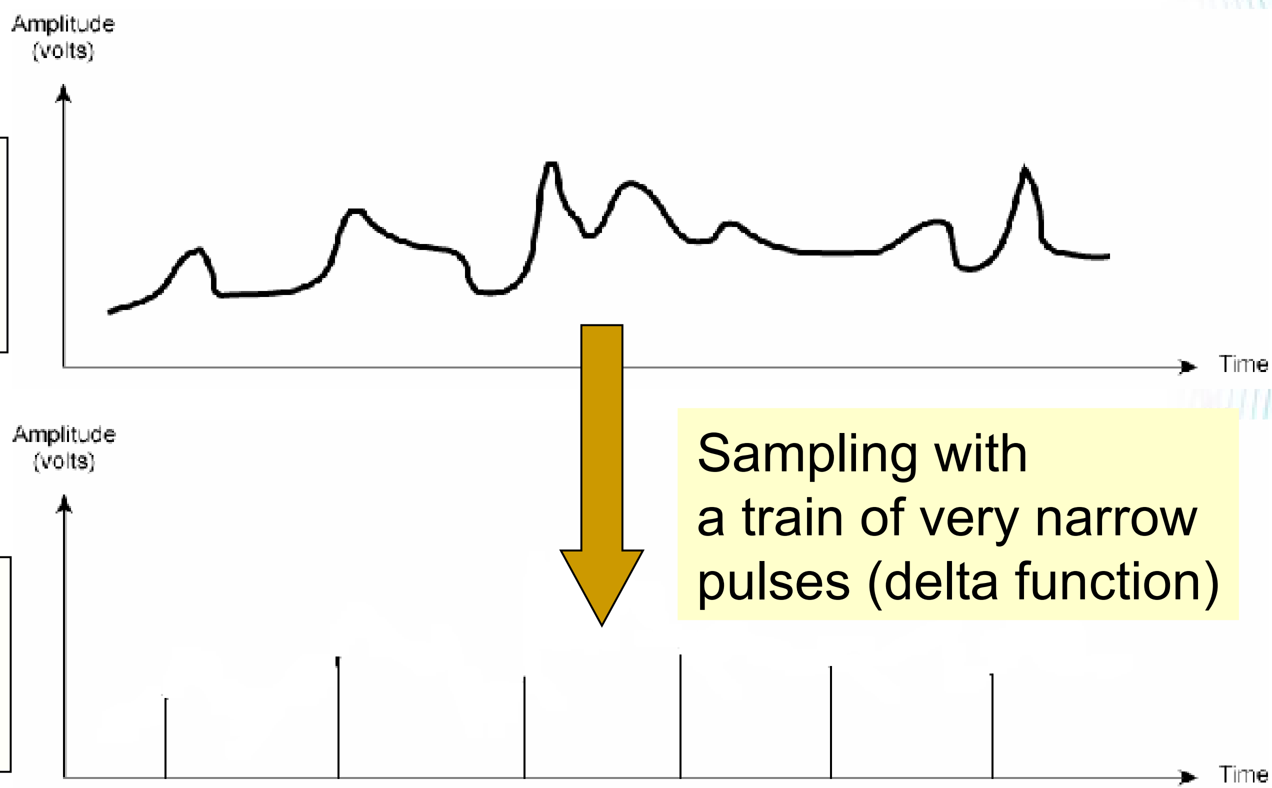
Availability of the signal over the horizontal axis (Time or Frequency)

Continuous:

Signal is defined at *all points* on the horizontal axis

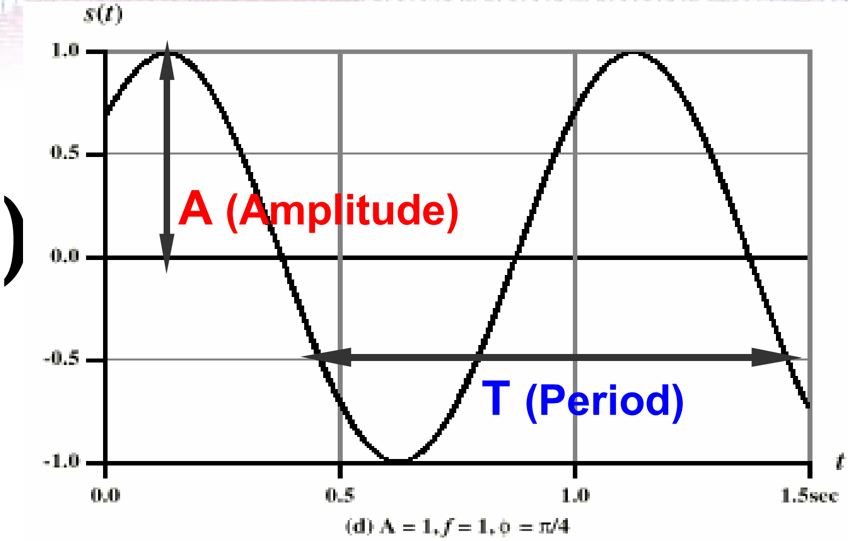
Discrete:

Signal is defined Only at *certain points* on the horizontal axis



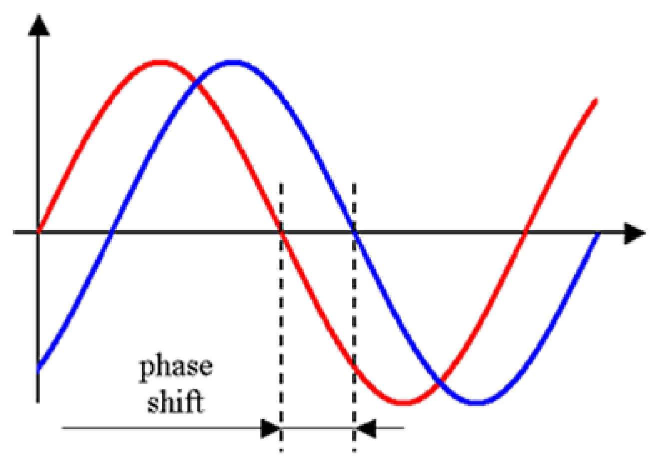
Sine Wave

s(t) = A sin(2πf t +Φ)

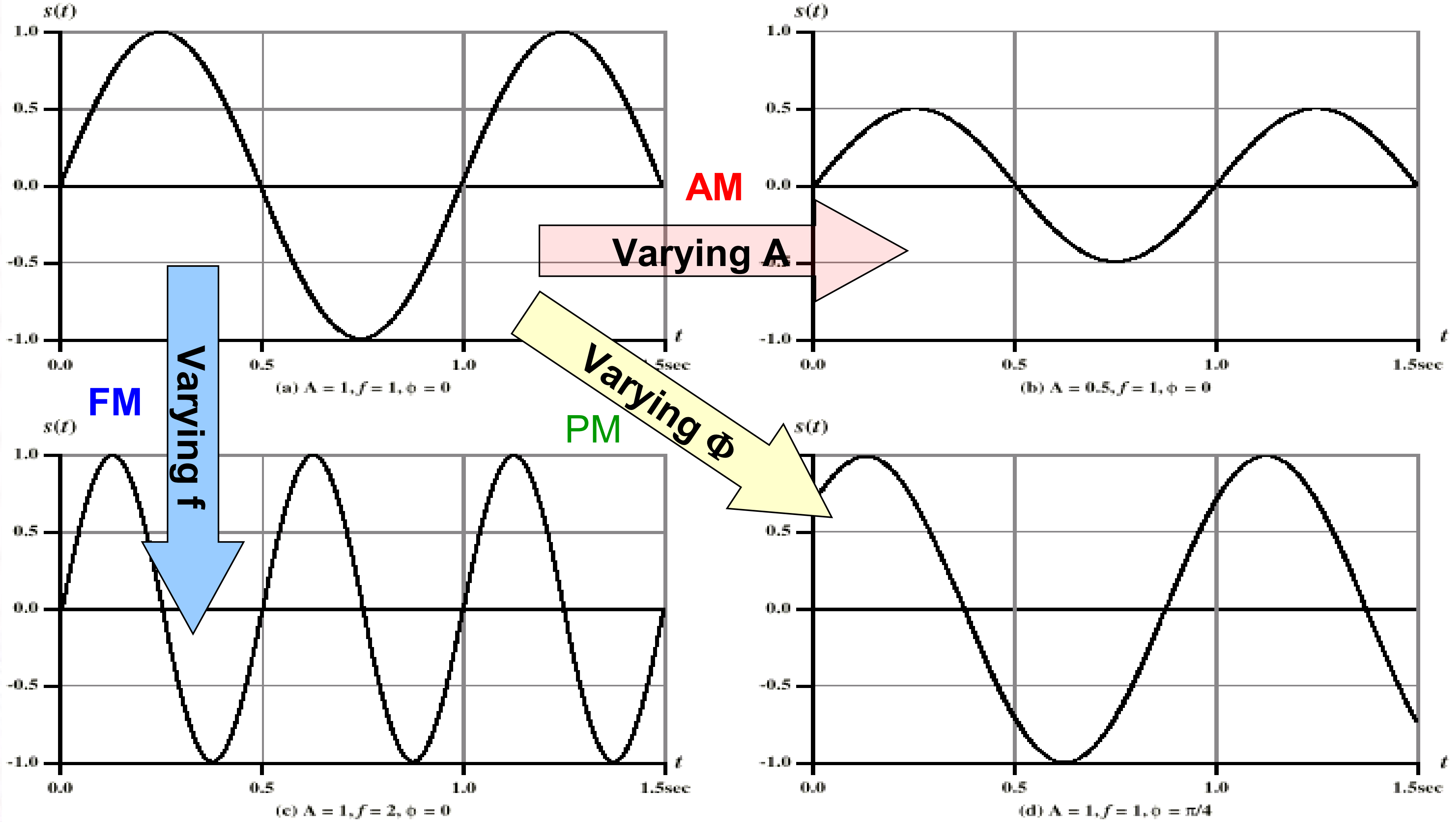


- Peak Amplitude (A)
  - Peak strength of signal, volts
- Repetition Frequency (f), Cycles/s = Hz
  - Measures how fast the signal varies with time
  - Number of cycles per second (Hz)
  - f = 1/T (secs/cycle) f = cycles/sec = Hz
- Angular Frequency (ω), Radians/s

  ω = radians per second = 2π f = 2π /T
- Temporal (time) Period, T = 1/f
- Phase Angle (ϕ), Radians
  - Determines relative position in time, radians (how to calculate?)



Varying one of the 3 parameters of a sine wave carrier $s(t)=A\ \sin(2\pi ft$



Sine Wave Travelling in the positive x direction: $s(t) = A \sin(k\,x - \omega\,t)$

k = Wave Number
$= 2\pi / \lambda$

ω = Angular Frequency
$= 2\pi f = 2\pi / T$

For point p on the wave:

Total phase at t = 0: $kx - \omega(0) = kx$

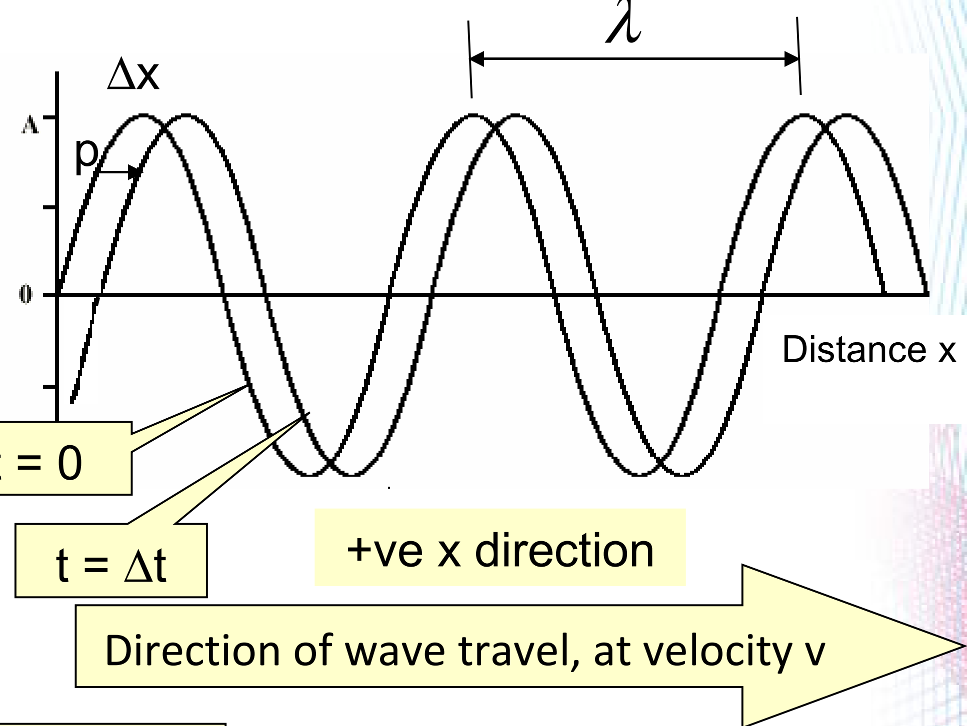
Total phase at t = Δt: $k(x + \Delta x) - \omega(\Delta t)$

Same total phase,
$\therefore kx = k(x + \Delta x) - \omega(\Delta t)$
$\therefore k\,\Delta x = \omega\,\Delta t$

Wave propagation velocity $v = \Delta x / \Delta t$
$v = \omega/k = \lambda/T = \lambda f$

→ Show that the wave
$s(t) = A \sin(k\,x + \omega\,t]$
travels in the negative x direction

λ = Spatial Period = Wavelength



$$v = \lambda f$$

V is constant for a given wave type (e.g. sound) and medium (e.g. air)

Wave Propagation Velocity, *v* m/s

- Constant for:
  - A any given wave type (e.g. electromagnetic, seismic, ultrasound, etc)
  - and a given propagation medium (air, water, optical fibre, etc)
- For all types of waves: $v = \lambda f$
- For a given wave type and medium (given v): higher frequencies correspond to shorter wavelengths and vice versa.

Electromagnetic waves
Shorter wavelength → Higher frequency

long wave radio (km), short wave radio (m), microwave (cm)… light (nm)

- For electromagnetic waves:
  - In free space, $v \approx$ speed of light in vacuum, $v \approx c = 3 \times 10^8$ m/s
  - Over other guided media (coaxial cable, optical fibre, twisted pairs): v is always lower than c

Wavelength, λ (meters)

- Is the **spatial period** of the wave:
  i.e. distance between two successive points in space on the wave propagation path where the wave has the same phase
- Also it is the distance traveled by the wave during one temporal (time) cycle:

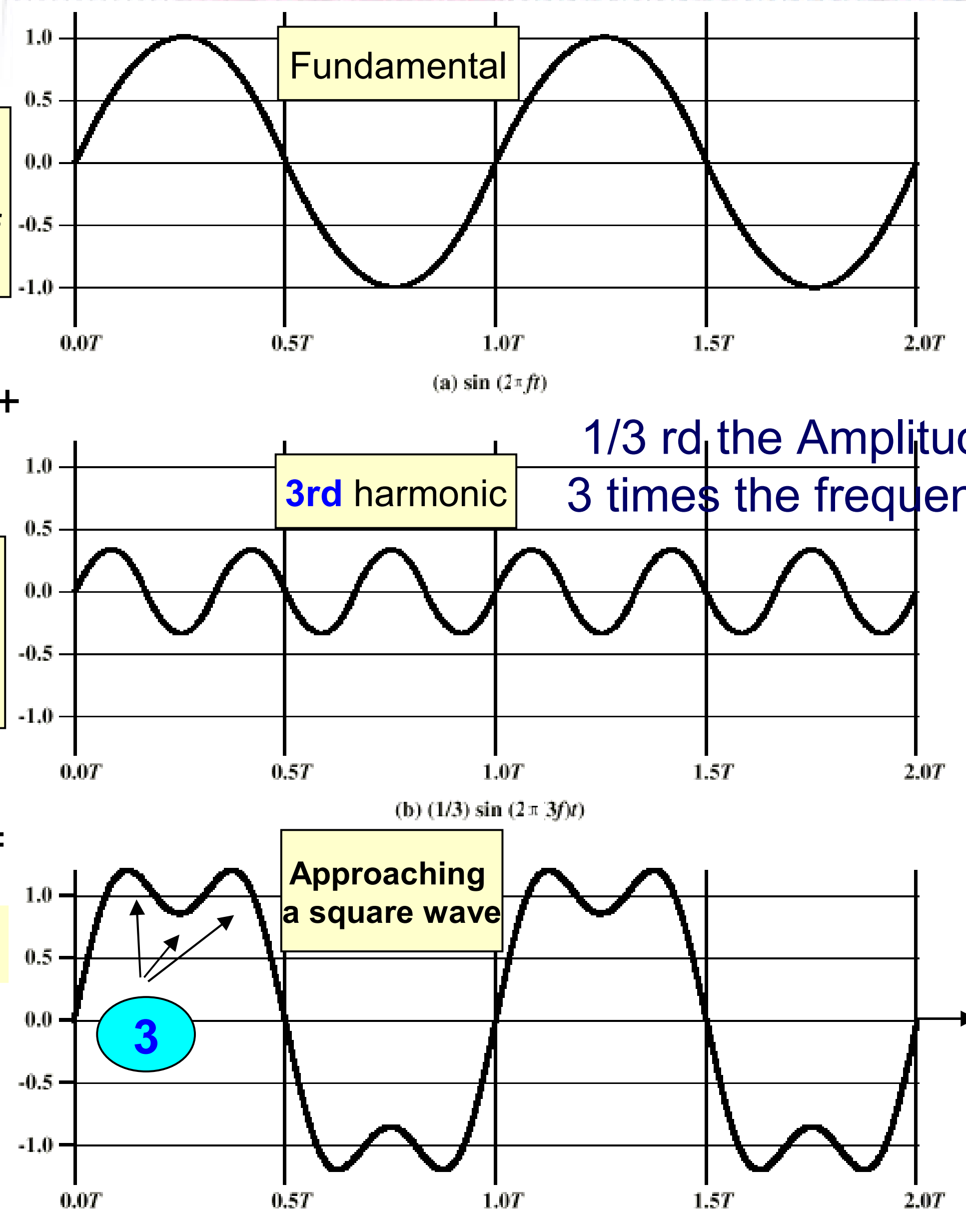
$$d_T = v\,T = (\lambda\, f)\,T = \lambda$$

Frequency Domain Concepts

- Response of systems to a sine wave is easy to analyse
- But signals we deal with in practice are not all sine waves, e.g. square waves in case of digital signal.
- Can we relate waves we deal with in practice to sine waves? YES!
- Fourier analysis shows that any signal can be treated as the sum of many sine wave components having different frequencies, amplitudes, and phases.
- This forms the basis for frequency domain analysis
- For a linear system, its response to a complex signal will be the sum of its responses to the individual sine wave components representing that signal.
- Dealing with functions in the frequency domain is simpler than in the time domain

Addition of Two frequency Components

A = 1*(4/π)
frequency = f



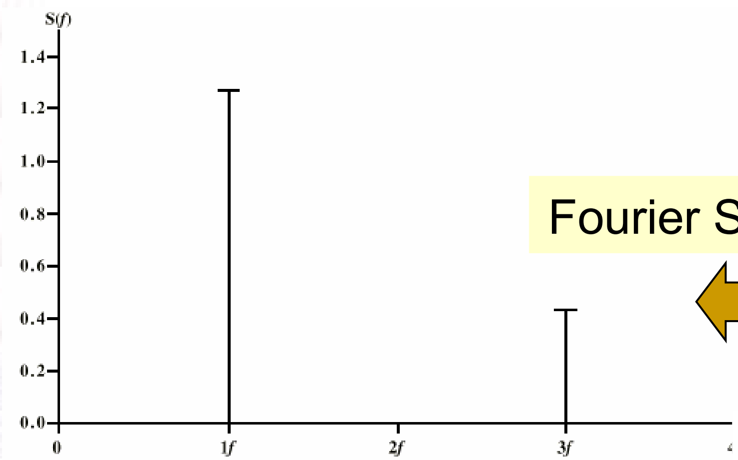
+

1/3 rd the Amplitude
3 times the frequency

A = (1/3)*(4/π)
frequency = 3f

=

Frequency Spectrum



Fourier Series

f

Frequency Domain: s(f) vs f

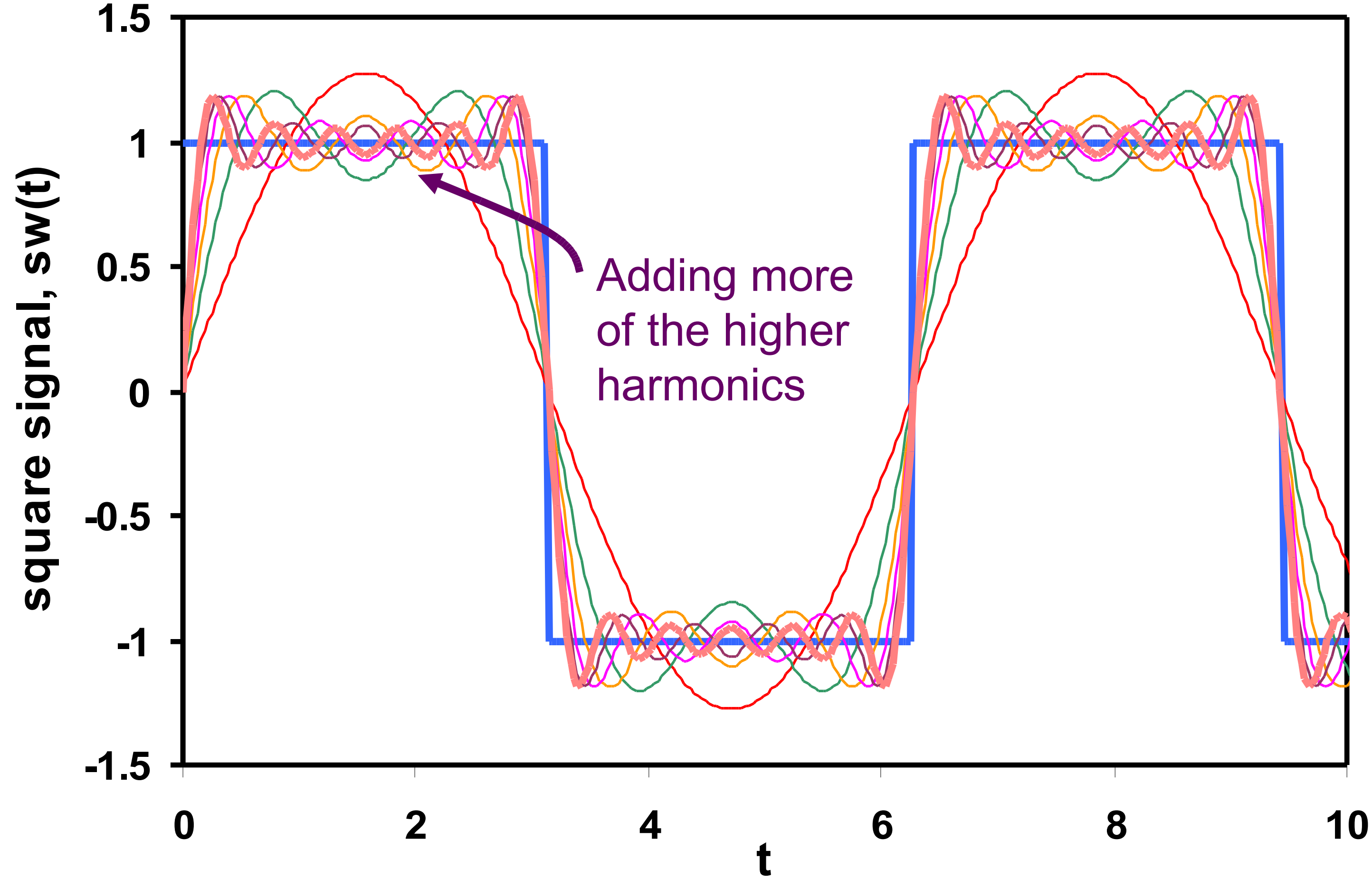
Discrete function in f

Fourier Series

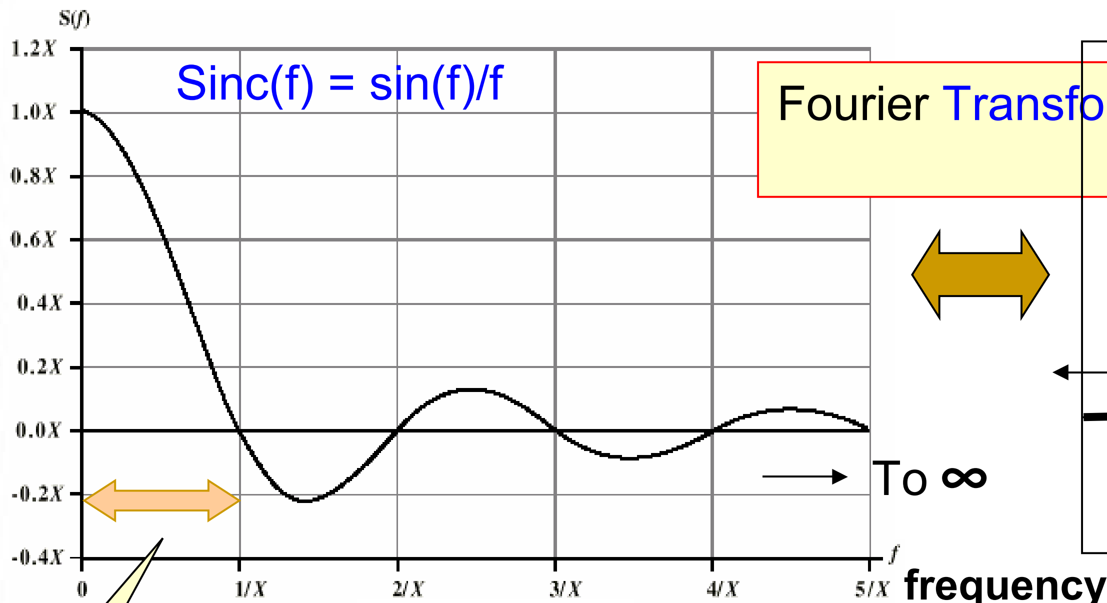
Time Domain: s(t) vs t

Periodic function in t

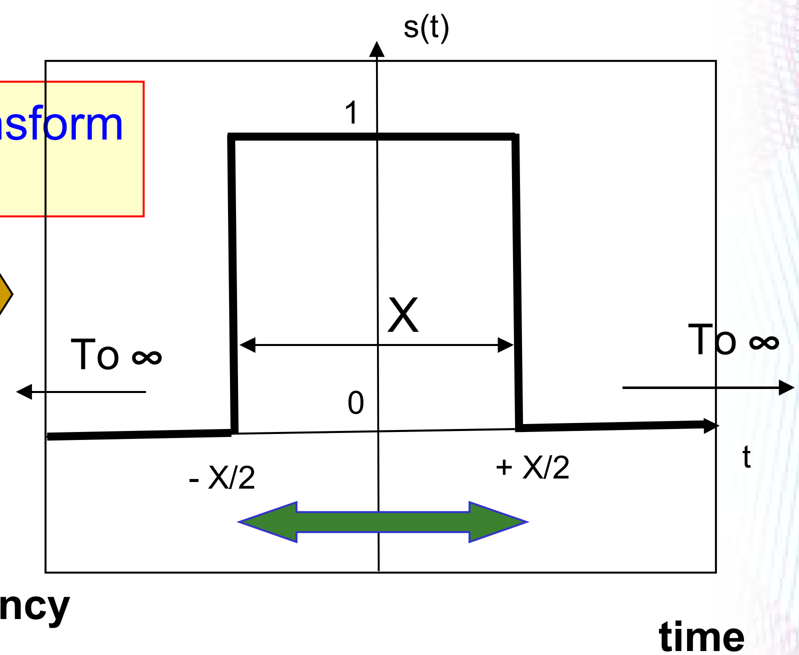
Asymptotically approaching a square wave by combining the fundamental + an infinite number of **odd** harmonics at prescribed amplitudes



Frequency Domain Representations: A single square pulse (Aperiodic signal)



Fourier Transform



1/X

Frequency Domain: S(f) vs f

Fourier Transform

Time Domain: s(t) vs t

Continuous Function in f

Aperiodic function in t

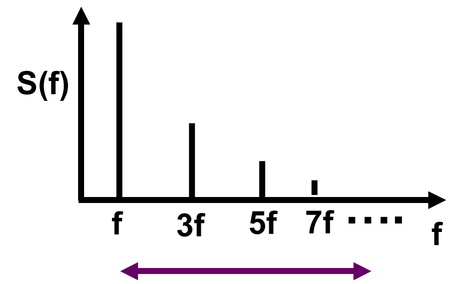
- What happens to the spectrum as the pulse gets broader …→ DC ?

- What happens to the spectrum as the pulse gets narrower …→ spike ?

Spectrum & Bandwidth of a signal

- Spectrum of a signal
  - Range of frequencies contained in a signal
- Absolute (theoretical) Bandwidth (BW):
  - Is the full width of spectrum = fmax- fmin
  - But in many situations, fmax = ∞!

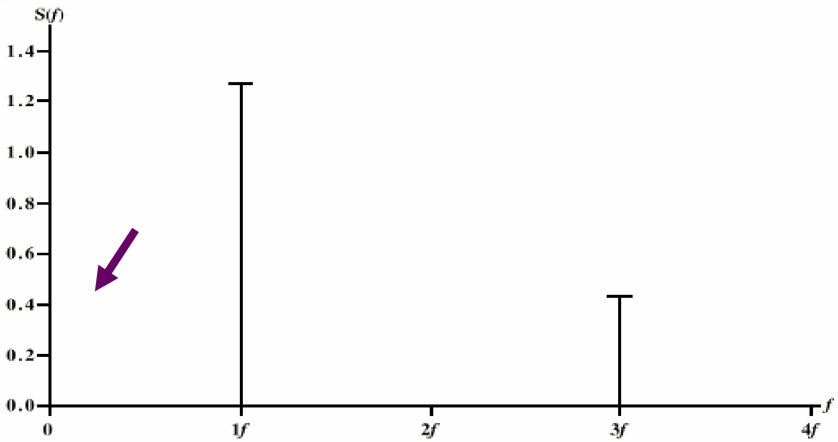
  (e.g. a square wave), so:



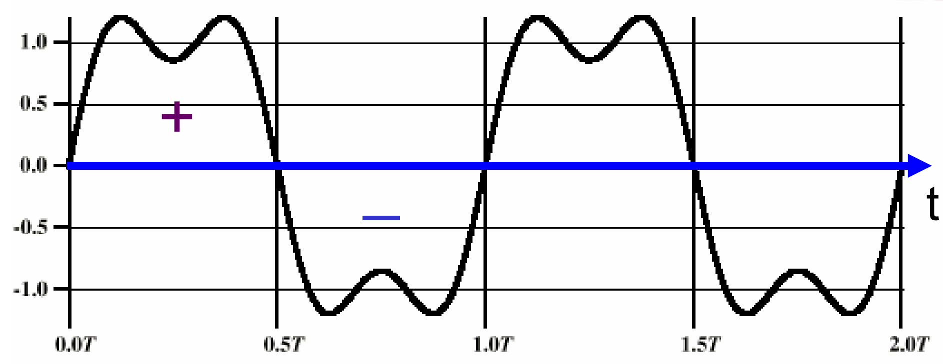
- Effective Bandwidth
  - Often called just *bandwidth*
  - Narrow band of frequencies containing *most* of the signal energy
  - Somewhat arbitrary: what is "most"?

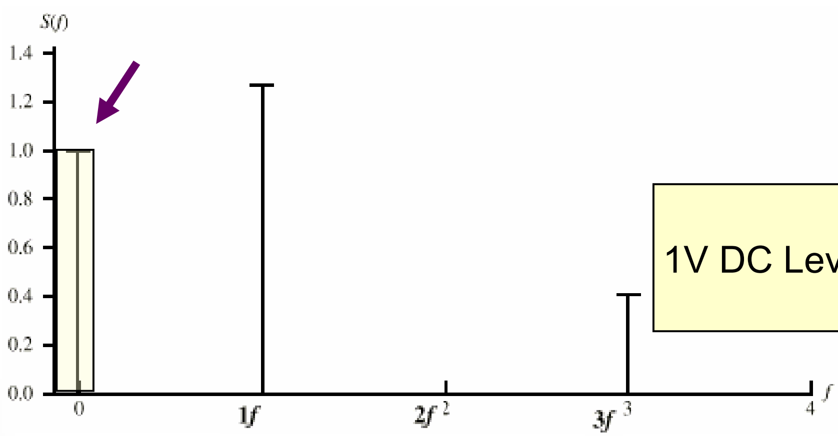
  → e.g. contains say 95% of the energy of the signal

Signals with a DC Component

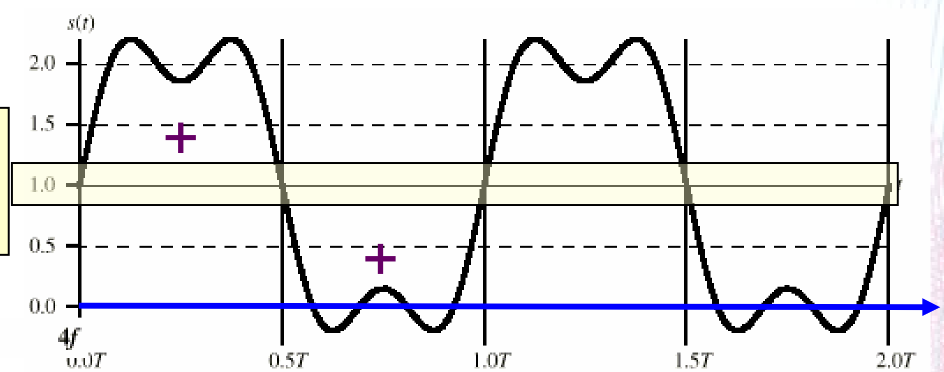


NO DC Component,
Signal average over a period = 0





1V DC Level

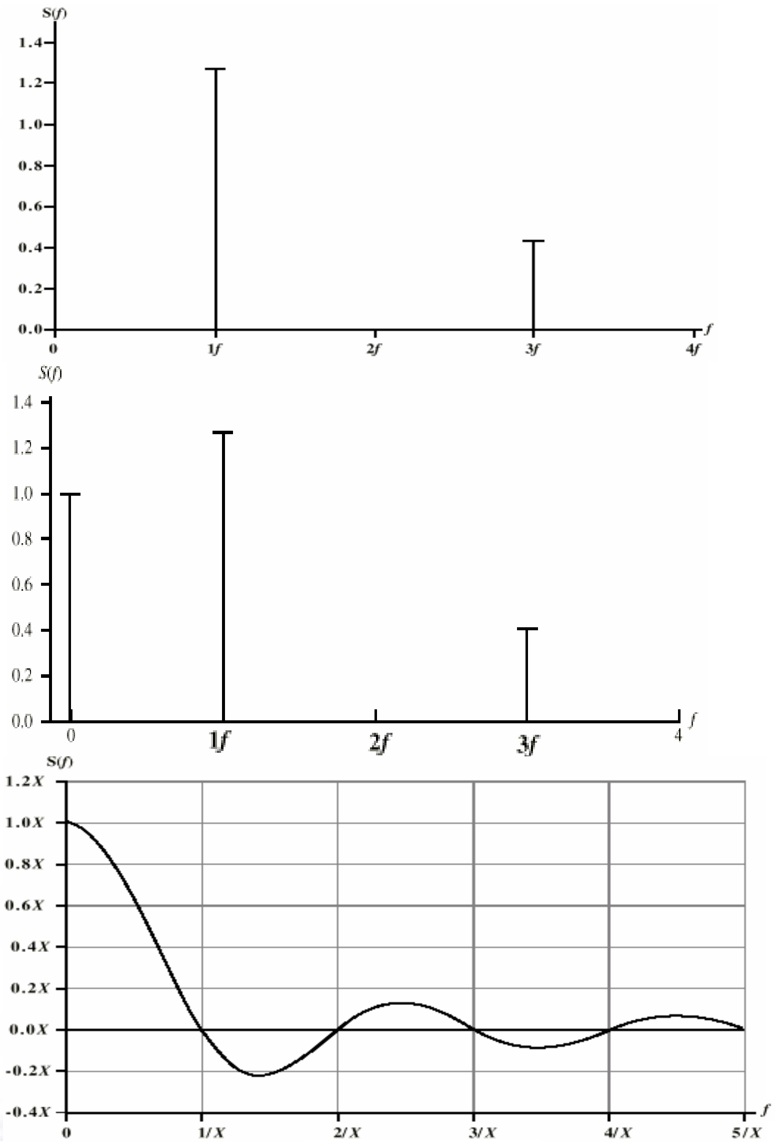


1V DC Component

DC Component:

→ is the component at zero frequency → Determines if fmin = 0 or not

Bandwidth for these signals:



fmin	*fmax*	Absolute BW	Effective BW
1f	3f	2f	2f
0	3f	3f	3f
0	∞	∞	1/X ?

= (fmax- fmin)

Bandwidth of a transmission system

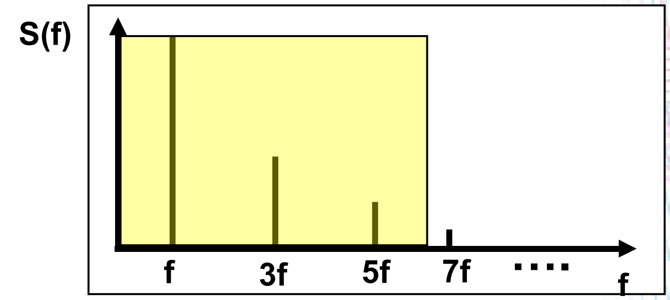
- Is the Range of signal frequencies that are adequately passed by the system
- Effectively, the transmission system (TX, medium, RX) acts as a filter
  - Poor transmission media, e.g. twisted pairs, have a narrow bandwidth
  - This effectively cuts off higher frequency signal components

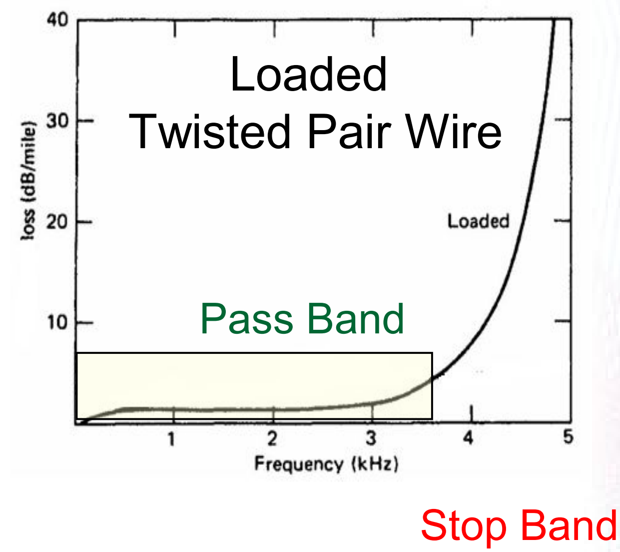
→ poor signal quality at receiver

→ limit the signal frequencies (Hz) that can be used for transmission

→ this limits the data rates that can be used (bps), examples:

  - Twisted pair: 4 KHz BW → 10's of Kbps
  - Optical fiber: 4 THz BW → 10's of Gbps





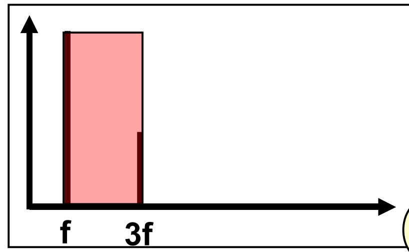
Limiting Effect of System Bandwidth

More difficult reception with smaller BW

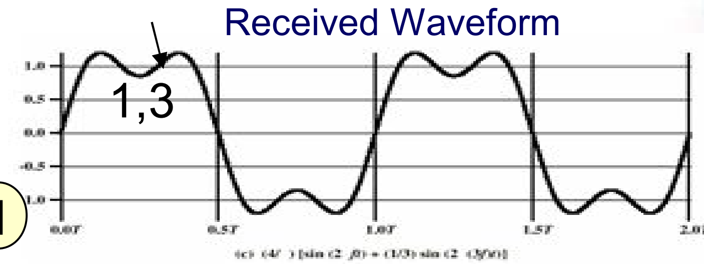
Varying System BW

Better reception requires larger BW

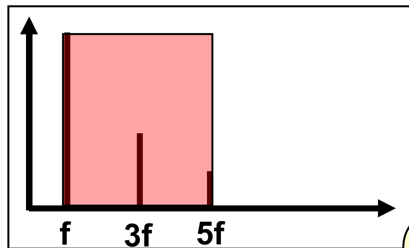
BW = 2f



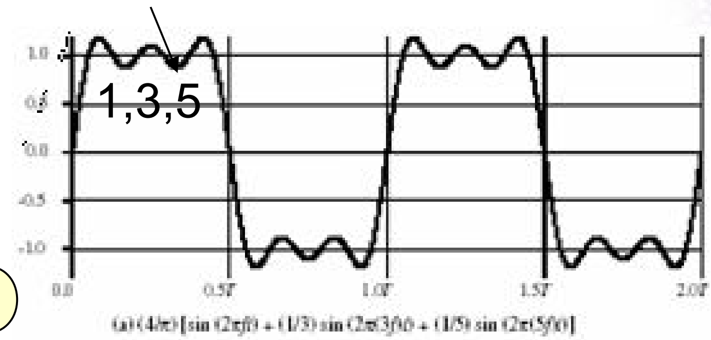
1



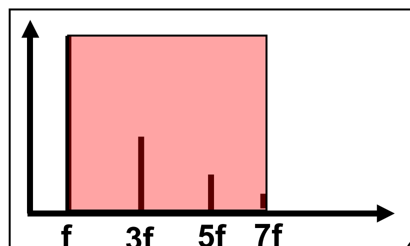
BW = 4f



2



BW = 6f



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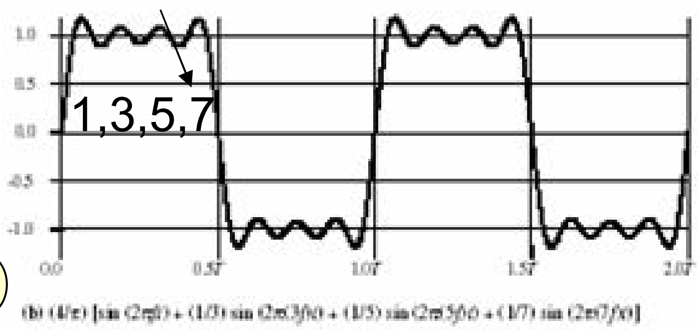
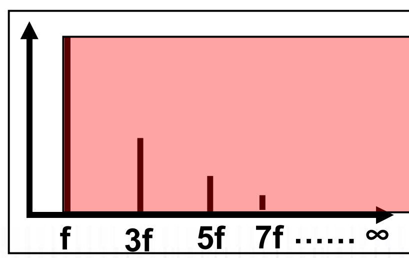
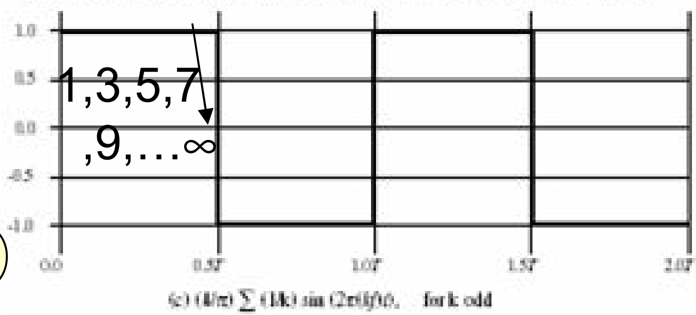


BW = ∞



4



$$s(t) = \frac{4}{\pi}\sum_{k\ \text{odd},\ k=1}^{\infty}\frac{1}{\text{k}}\sin(2\pi kft)$$

Fourier Series for a Square Wave

System Bandwidth and Achievable Data Rates

- Any transmission system supports only a limited range of frequencies (bandwidth) for satisfactory transmission
- For example, this bandwidth is largest for expensive optical fibres and smallest for cheap twisted pair wires
- So, bandwidth is money → Economize in its use
- Limited system bandwidth degrades higher frequency components of the signal transmitted

⇒ poorer received waveforms

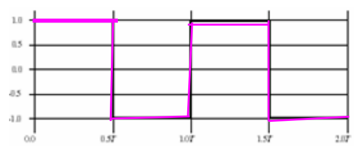
⇒ more difficult to interpret the signal at the receiver (especially with noise) ⇒ Data Errors

- More degradation occurs when higher data rates are used (signal will have higher frequency components)

Fourier Analysis

Signals in Time

Periodic



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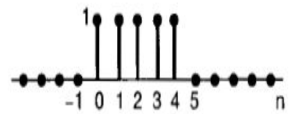
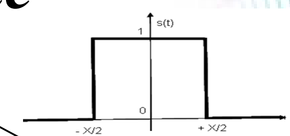
Aperiodic





Discrete	***Continuous***	***Discrete***	***Continuous***
DFS	**FS**	**DFT**	**FT**

Use Fourier Series (FS)

Use Fourier Transform (FT)

FS : Fourier Series
DFS : Discrete Fourier Series
FT : Fourier Transform
DFT : Discrete Fourier Transform

Fourier Series for periodic signals

Any periodic signal x(t) of period T and repetition frequency f_0 (f_0 = 1/T) can be represented as an infinite sum of sinusoids of different frequencies and amplitudes – its Fourier Series expressed in two forms:

1. The sine/cosine form:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t) \right]$$

Frequencies are *multiples* of the fundamental frequency f_0

DC Component

f_0 = fundamental frequency = 1/T

Where:

$$A_0 = \frac{2}{T} \int_0^T x(t)\,dt$$

$$A_n = \frac{2}{T} \int_0^T x(t) \cos(2\pi n f_0 t)\,dt \quad = f(n)$$

Two components at each frequency

All integrals are over one period only

If A_0 is not 0, *x(t)* has a DC component

$$B_n = \frac{2}{T} \int_0^T x(t) \sin(2\pi n f_0 t)\,dt \quad = f'(n)$$

Fourier Series for periodic signals

2. The Amplitude/Phase Form

- Previous form had two components at each frequency (sine, cosine i.e. in quadrature) : A_n, B_n coefficients
- The equivalent Amplitude-Phase representation has only one component at each frequency: C_n, θ_n
- Derived from the previous form using trigonometry:

cos (a) cos (b) - sin (a) sin (b) = cos [a +b]

Now we have Only one component at each frequency nf_0

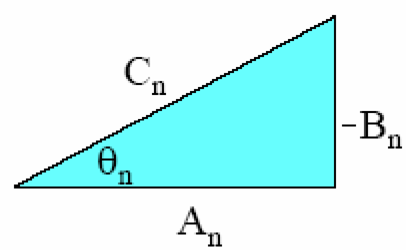
$$x(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \left[C_n \cos(2\pi n f_0 t + \theta_n) \right]$$

Now components have different amplitudes, frequencies, and phases

The C's and θ's are obtained from the previous A's and B's using the equations:

$$C_0 = A_0 \qquad C_n = \sqrt{A_n^2 + B_n^2}$$

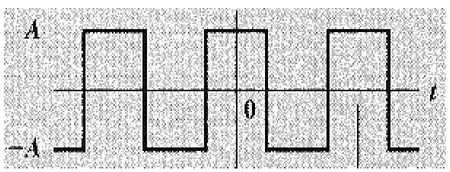
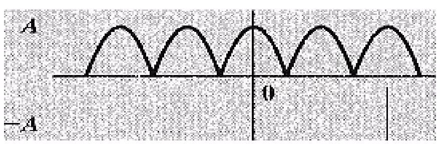
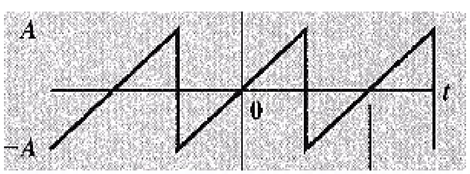
$$\theta_n = \tan^{-1}\left(\frac{-B_n}{A_n} \right)$$



Fourier Series: General Observations

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)\right]$$ ← Fourier Series Expansion

Function: $x(t)$; DC: $\frac{A_0}{2}$; Even Function: $\cos(2\pi n f_0 t)$; Odd Function: $\sin(2\pi n f_0 t)$

Function		Series
No DC 	$\frac{1}{T}\int_0^T x(t)dt = 0$	$A_0 =$ 2 DC = 0
Even Function 	x(t) = x(-t) Symmetric about Y axis	$B_n = 0$; for all n
Odd Function 	x(t) = - x(-t) Symmetric about the origin	$A_n = 0$; for all n

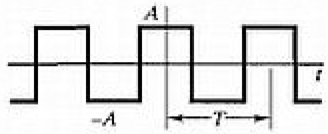
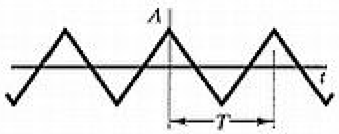
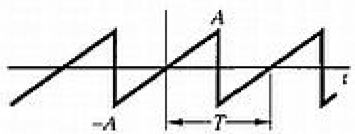
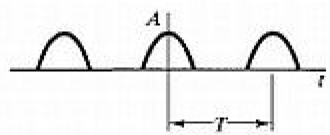
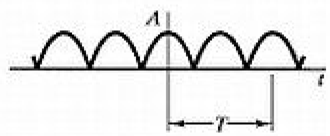
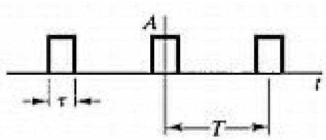
Square wave 	$\frac{4A}{\pi}(\cos \omega_1 t - \frac{1}{3}\cos 3\omega_1 t + \frac{1}{5}\cos 5\omega_1 t - \frac{1}{7}\cos 7\omega_1 t + \ldots)$
Triangular wave 	$\frac{8A}{\pi^2}(\cos \omega_1 t + \frac{1}{9}\cos 3\omega_1 t + \frac{1}{25}\cos 5\omega_1 t + \ldots)$
Sawtooth wave 	$\frac{2A}{\pi}(\sin \omega_1 t - \frac{1}{2}\sin 2\omega_1 t + \frac{1}{3}\sin 3\omega_1 t - \frac{1}{4}\sin 4\omega_1 t + \ldots)$
Half-wave rectified cosine 	$\frac{A}{\pi}(1 + \pi\cos \omega_1 t + \frac{2}{3}\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_1 t + \frac{2}{35}\cos 6\omega_1 t \ldots (-1)^{\frac{n}{2}+1}\frac{2}{n^2-1}\cos n\omega_1 t + \ldots)$ n even
Full-wave rectified cosine 	$\frac{2A}{\pi}(1 + 2\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_1 t + \frac{2}{35}\cos 6\omega_1 t - \ldots (-1)^{\frac{n}{2}+1}\frac{2}{n^2-1}\cos n\omega_1 t + \ldots)$ n even
Pulse train 	$Ad[1 + 2(\frac{\sin \pi d}{\pi d}\cos \omega_1 t + \frac{\sin 2\pi d}{2\pi d}\cos 2\omega_1 t + \frac{\sin 3\pi d}{3\pi d}\cos 3\omega_1 t + \ldots)]$ $d = \tau/T$

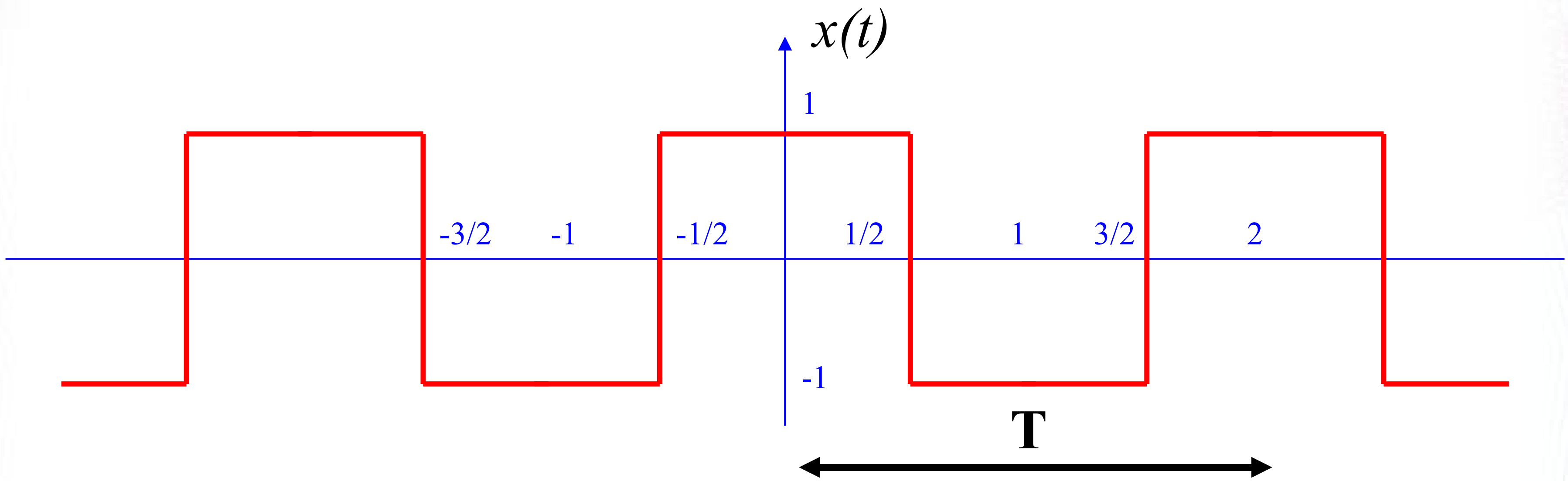
Figure 2.17 from

Stallings W. (1997) *Data and Computer Communications,* 5th Edition

which you can download on my site in the intr@web server at

http://intraweb/~rh/download/networking.html

Fourier Series Example



Note: (1) $x(-t)=x(t) \Rightarrow x(t)$ *is an even function*

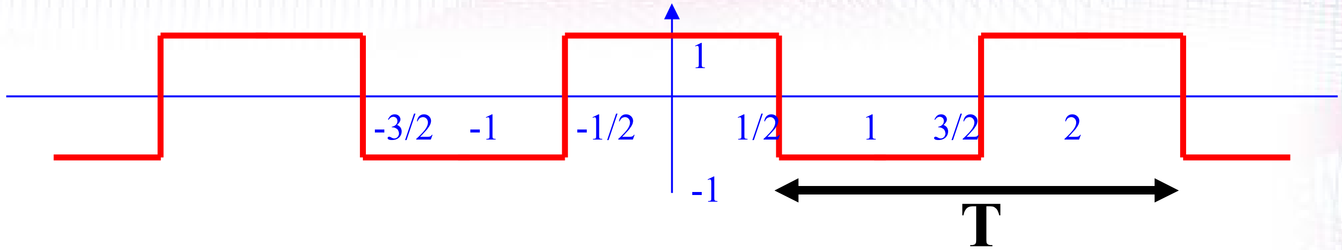
(2) $f_0 = 1/T = ½\ Hz$

$$A_0 = \frac{2}{T}\int_0^T x(t)dt = \frac{2}{2}\int_0^2 x(t)dt = 2\int_0^1 x(t)dt = 2[\int_0^{1/2} 1dt + \int_{1/2}^{1} -1]dt$$

$$= 2\{[t]_0^{1/2} - [t]_{1/2}^{1}\} = 0$$

Note: A0 by definition is 2 x the DC content

Cont…



$$A_n = \frac{2}{T}\int_0^T x(t)\cos(2\pi n f_0 t)dt = \frac{4}{T}\int_0^{T/2} x(t)\cos(2\pi n f_0 t)dt = 2\int_0^1 x(t)\cos(2\pi n f_0 t)dt$$

$$= 2\int_0^{1/2}\cos(2\pi n f_0 t)dt + 2\int_{1/2}^{1} -\cos(2\pi n f_0 t)dt$$

f0 =1/2

= 0 for n even

$$= \frac{4}{n\pi}\sin\frac{n\pi}{2}$$ **for n odd**

a function of n only

$$B_n = \frac{2}{T}\int_0^T x(t)\sin(2\pi n f_0 t)dt = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(2\pi n f_0 t)dt$$

Take period as –T/2 to +T/2

$$= \frac{2}{T}\int_{-T/2}^{0} x(t)\sin(2\pi n f_0 t)dt + \frac{2}{T}\int_0^{T/2} x(t)\sin(2\pi n f_0 t)dt$$

in the first integral:

→ Replace *t* by –*t*

→ Swap limits

$$= -\frac{2}{T}\int_0^{T/2} x(-t)\sin(-2\pi n f_0 t)d(-t) + \frac{2}{T}\int_0^{T/2} x(t)\sin(2\pi n f_0 t)dt = 0$$

$\sin(-2\pi n f_0 t) \to -\sin(2\pi n f\, t)$, $\; d(-t) \to -dt$

$x(-t) \to$ *x(t)*, since *x(t)* is an even function

Then *Bn* = 0 *for all n*

Cont…

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)\right]$$

$f_0 = ½$, so $2\pi f_0 = \pi$

$A_0 = 0$,

$B_n = 0$ for all n,

$A_n = 0$ for n even: 2, 4, …

$= (4/n\pi) \sin(n\pi/2)$ for n odd: 1, 3, …

$$x(t) = \sum_{n=1,odd}^{\infty} \frac{4}{n\pi} \sin\frac{n\pi}{2} \cos n\pi t$$

Amplitudes, n odd

Original x(t) is an **even** function!

$$x(t) = \frac{4}{\pi}\cos\pi t - \frac{4}{3\pi}\cos 3\pi t + \frac{4}{5\pi}\cos 5\pi t - \frac{4}{7\pi}\cos 7\pi t + \ldots$$

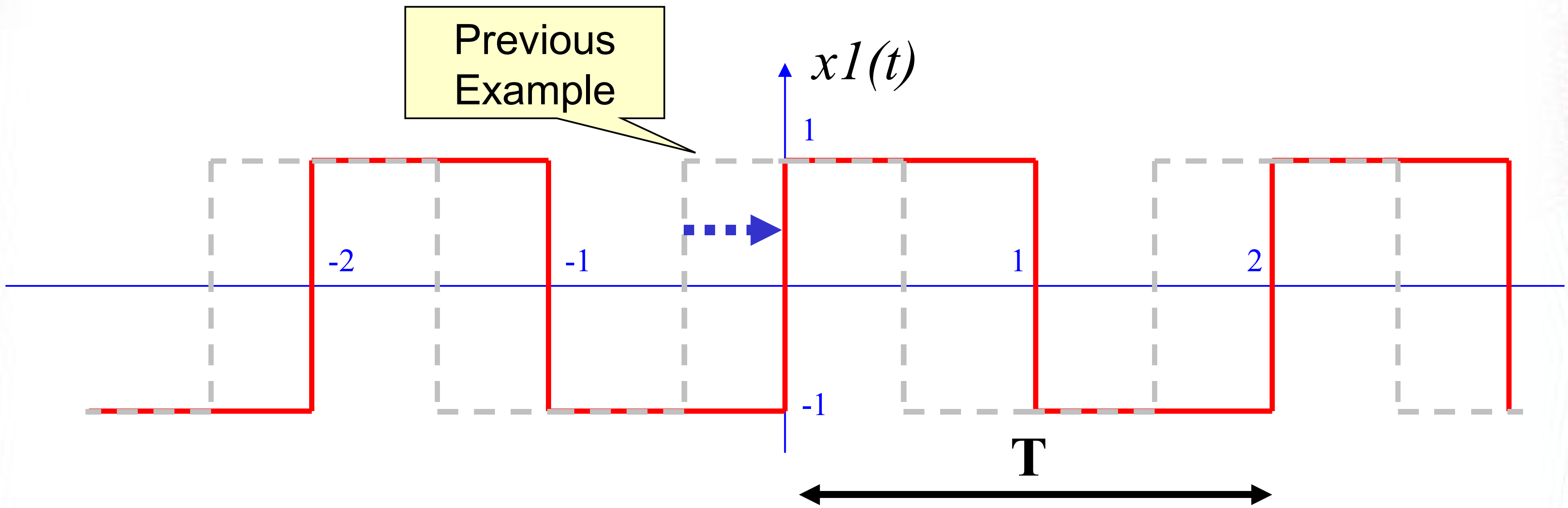
$$x(t) = \frac{4}{\pi}\left[\cos\pi t - \frac{1}{3}\cos 3\pi t + \frac{1}{5}\cos 5\pi t - \frac{1}{7}\cos 7\pi t + \ldots\right]$$

Cosine is an even function

2 π (1/2) t — 2 π 3 (1/2) t

f0 = ½ → Fundamental — 3rd Harmonic

Another Example



Note that $x1(-t) = -x1(t) \Rightarrow$ *so,* $x(t)$ *is an odd function*

Also, $x1(t)=x(t-1/2)$ This waveform is the previous waveform shifted right by 1/2

$$x1(t) = \frac{4}{\pi}\left[\cos\pi\left(t-\frac{1}{2}\right) - \frac{1}{3}\cos 3\pi\left(t-\frac{1}{2}\right) + \frac{1}{5}\cos 5\pi\left(t-\frac{1}{2}\right) - \frac{1}{7}\cos 7\pi\left(t-\frac{1}{2}\right) + \ldots\right]$$

Another Example, Contd…

$$x1(t) = \frac{4}{\pi}\left[\cos\left(\pi t - \frac{\pi}{2}\right) - \frac{1}{3}\cos\left(3\pi t - \frac{3\pi}{2}\right) + \frac{1}{5}\cos\left(5\pi t - \frac{5\pi}{2}\right) - \frac{1}{7}\cos\left(7\pi t - \frac{7\pi}{2}\right)\right] + \ldots$$

$$x1(t) = \frac{4}{\pi}\left[\sin\pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \frac{1}{7}\sin 7\pi t + \ldots\right]$$

Sine is an odd function

$$x1(t) = \frac{4}{\pi}\sum_{k\ \text{odd},\ k=1}^{\infty}\frac{1}{k}\sin(2\pi k f_0 t)$$

As given before for the square wave on slide 30.

Because:

$$\cos\left(\pi t - \frac{\pi}{2}\right) = \sin\pi t \qquad \cos\left(3\pi t - \frac{3\pi}{2}\right) = -\sin 3\pi t$$

$$\cos\left(5\pi t - \frac{5\pi}{2}\right) = \sin 5\pi t \qquad \cos\left(7\pi t - \frac{7\pi}{2}\right) = -\sin 7\pi t$$

etc.

Fourier Transform

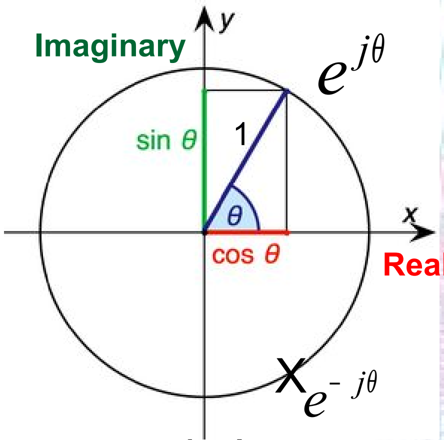
- For continuous aperiodic (non-periodic) signals in time, the spectrum consists of a continuum of frequencies (not discrete components)
  - This spectrum is defined by the Fourier Transform
  - For a signal x(t) and a corresponding spectrum X(*f*), the following pair of relations hold

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \qquad x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df$$

T/2 → ∞ nf_0 → f

Forward FT (from time to frequency) **Inverse** FT (from frequency to time

$$e^{-j\theta} = \cos\theta - j\sin\theta \qquad e^{j\theta} = \cos\theta + j\sin\theta$$



→ Obtain sin and cos in terms of $e^{j\theta}$ *and* $e^{-j\theta}$

- X(f) is always complex (Has both real & Imaginary parts), even for x(t) real.

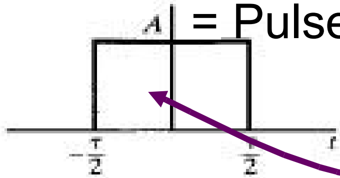
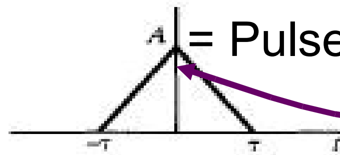
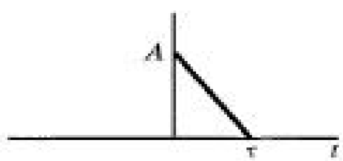
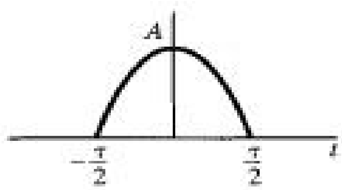
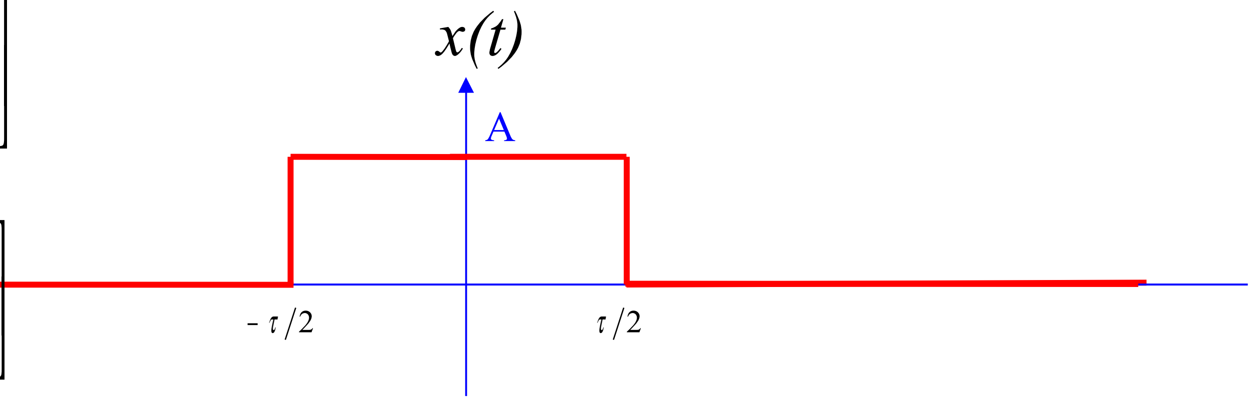
(non-periodic in time) Signal $x(t)$		(Continuous in Frequency) Spectrum $X(f)$	
Rectangular pulse		$A\tau \frac{\sin \pi f\tau}{\pi f\tau}$	Sinc function
Triangular pulse		$A\tau\left(\frac{\sin \pi f\tau}{\pi f\tau}\right)^2$	$Sinc^2$ function
Sawtooth pulse		$\frac{jA}{2\pi f}\left[\frac{\sin \pi f\tau}{\pi f\tau} e^{-j\pi f\tau} - 1\right]$	
Cosine pulse		$\frac{2A\tau}{\pi} \frac{\cos \pi f\tau}{1-4f^2\tau^2}$	

Fig. 2.18 in Textbook

Fourier Transform Example

$$\sin\theta = \left[\frac{e^{j\theta} - e^{-j\theta}}{2j}\right]$$

$$\cos\theta = \left[\frac{e^{j\theta} + e^{-j\theta}}{2}\right]$$



$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt$$

$$X(f) = \int_{-\tau/2}^{\tau/2} A\, e^{-j2\pi ft} dt = -\frac{A}{j2\pi f} e^{-j2\pi ft}\Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{\pi f}\left[\frac{e^{j2\pi f\tau/2} - e^{-j2\pi f\tau/2}}{2j}\right] = \frac{A}{\pi f}\left(\frac{\pi f\tau}{1}\right)\left(\frac{\sin(\pi f\tau)}{\pi f\tau}\right) = A\tau\left(\frac{\sin(\pi f\tau)}{\pi f\tau}\right)$$

Sin (x) / x i.e. “sinc” function

Area of pulse In time domain

Fourier Transform Example, contd.

$$X(f) = A\tau \frac{\sin(\pi f \tau)}{\pi f \tau}$$

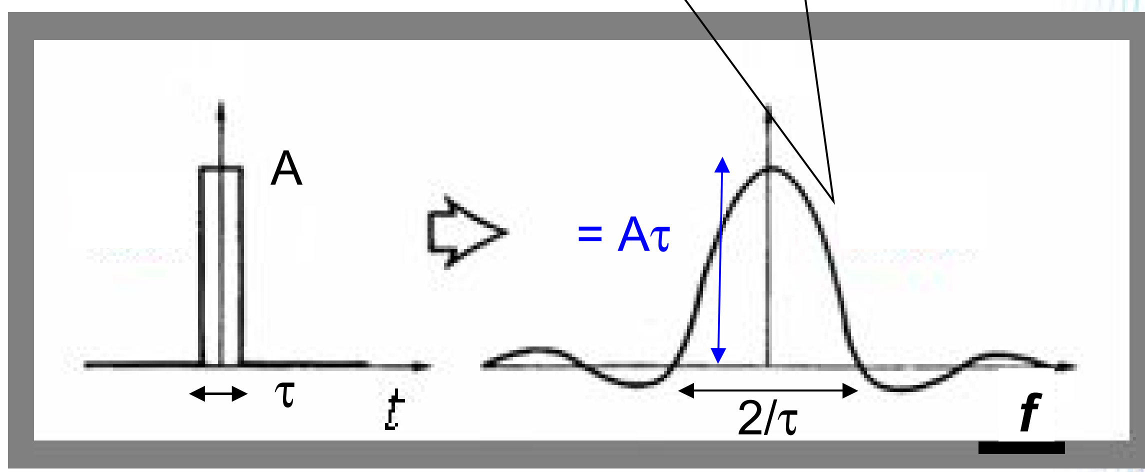
Constant → $A\tau$

Sin (x) / x
"sinc" function

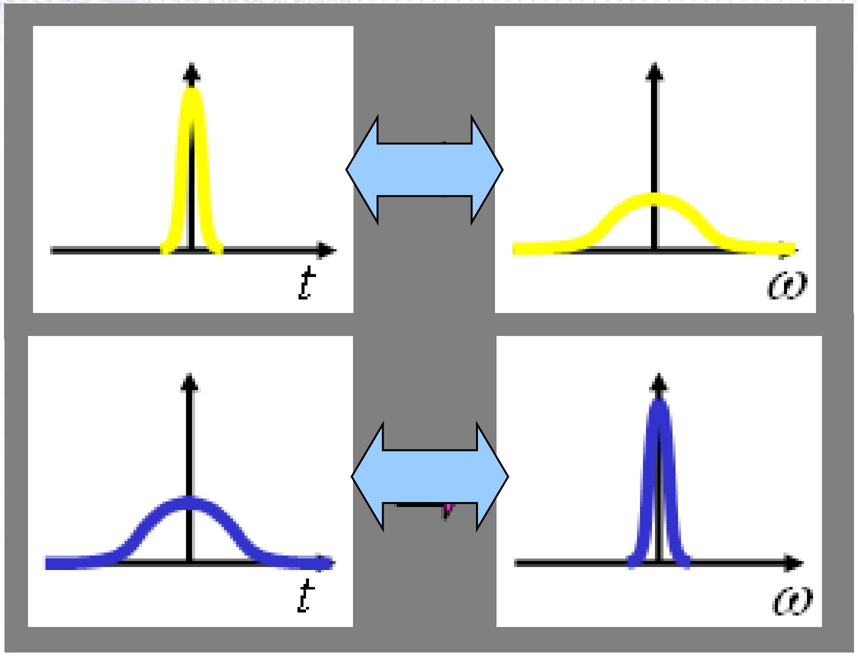
$\text{Lim}_{x \to 0}$ (sin x)/x = $(\cos x)_{x=0}/1 = 1$

First zero in the Frequency spectrum:

$\sin \pi f \tau = 0$
$\pi f \tau = \pi$
$f = 1/\tau$



Study the effect of the pulse width τ

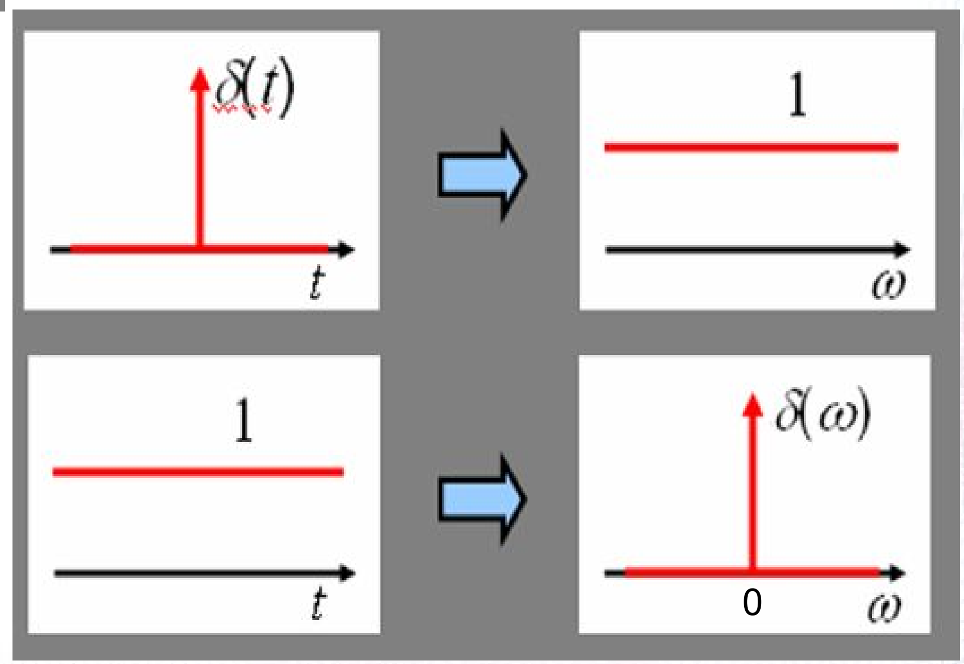


The narrower a function is in one domain, the wider its transform is in the other domain

Time

Frequency

A very sharp Pulse (0 width)



The Extreme Cases

DC

Power Spectral Density & Bandwidth

- Power spectral density (PSD) describes the *distribution* of the power content of a signal as a function of frequency
- Absolute bandwidth of any time-limited signal is infinite

  For periodic signals → Fourier Series, Discrete Spectrum:
- Luckily, strength of the nth harmonic component gets smaller with larger n
- i.e. most of the signal power will be concentrated in a finite band of lower frequencies
- Effective bandwidth is the width of the spectrum portion containing most (e.g. 95%) of the total signal power
- We first determine this total signal power for the signal in the time domain

Signal Power in the time domain

- Signal is specified as a function s(t) representing signal voltage or current
- Assuming resistance R = 1 Ω,

Instantaneous signal power (t) = $v(t)^2/1 = i(t)^2 * 1 = |s(t)|^2$

- Total signal power (i.e. due to all its frequency components) is taken as the **average** of the instantaneous signal power over a given interval of time = constant

$$P_{Total} = \frac{1}{t2 - t1} \int_{t1}^{t2} |x(t)|^2 \, dt$$

- For periodic signals, this averaging is usually taken over one period, i.e.

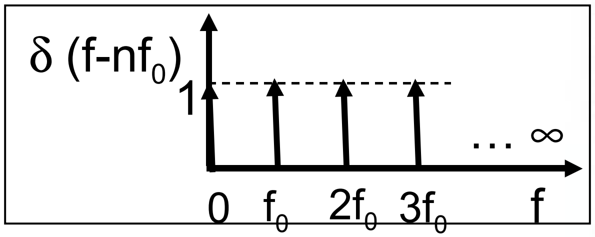
$$P_{Total} = \frac{1}{T} \int_{0}^{T} |s(t)|^2 \, dt$$

- This measure in the time domain gives the *total* signal power (i.e. in **all** its harmonic components up to f = ∞)
- Effective BW is then determined as that containing a specified portion (percentage) of this total signal power

Signal Power in the Frequency Domain: Periodic signals

- For periodic signal we have a discrete spectrum (the F Series):

$$x(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \left[C_n \cos(2\pi n f_0 t + \theta_n) \right]$$



  - For a DC component, Power = V_{dc}^2
  - For AC components Power = $V_{rms}^2 = \frac{1}{2} V_{peak}^2$ **(use equation on previous slide)**

- Power spectral density (PSD) is a discrete function of frequency:

$$PSD = \frac{C_0^2}{4} \times \delta(f) + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2 \times \delta(f - n f_o)$$

Where δ(*f*) is the Dirac delta function: $\delta(f) = \begin{cases} 1 & f = 0 \\ 0 & f \neq 0 \end{cases}$

- Total signal power (watts) up to the *j* th harmonic is:

$$P_{\text{Upto the } j \text{ th Component}} = \frac{1}{4} C_0^2 + \frac{1}{2} \sum_{n=1}^{j} C_n^2$$

(A quantity, *summation* of PSD components- not a function of a frequency)

Example

- Consider the following signal

$$x(t) = \left[1\sin\pi t \boxed{\pm} \frac{1}{3}\sin 3\pi t \pm \frac{1}{5}\sin 5\pi t \pm \frac{1}{7}\sin 7\pi t \right]$$

(No DC)

- The PSD is: (A function of Frequency)

$$PSD(f) = \frac{1}{2}[1^2\delta(f-0.5) + (\frac{1}{3})^2\delta(f-1.5) + (\frac{1}{5})^2\delta(f-2.5) + (\frac{1}{7})^2\delta(f-3.5)]$$

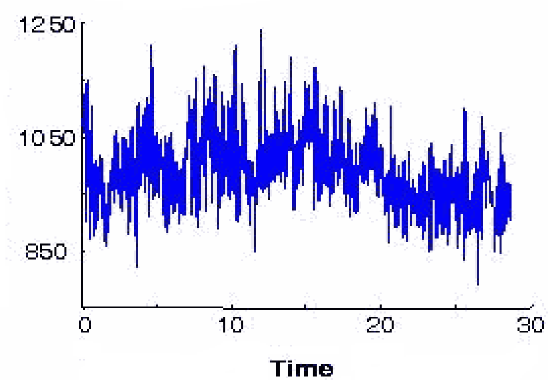
- The signal power is: (A quantity)

All positive components

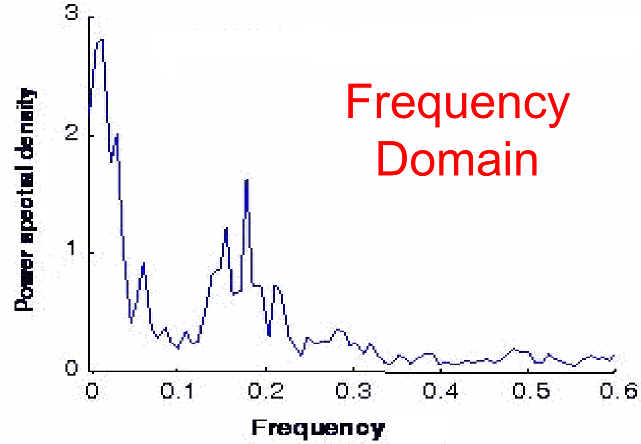
$$Power]_{1,3,5,7th\ harmonics} = \frac{1}{2}\left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49}\right] = 0.586\ \text{watt}$$

Signal Power in the Frequency Domain: Aperiodic signals

Time Domain



S(f) Watts/Hz



Frequency Domain

- Continuous (not discrete) frequency spectrum
- PSD (Power spectrum density) function, in *Watts/Hz,* is a *continuous* function of frequency: *S(f),*
- Total signal power contained in the frequency band $f_1 < f < f_2$ (in Watts) is given by:

(Integration, instead of summation, over the frequency range)

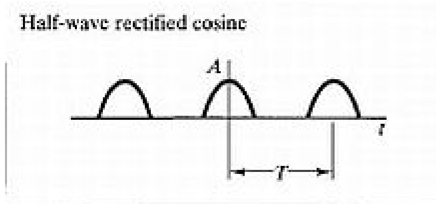
Components exist in both negative and positive frequencies

Watts/Hz

$$P = \boxed{2}\int_{f_1}^{f_2} S(f)\,df$$

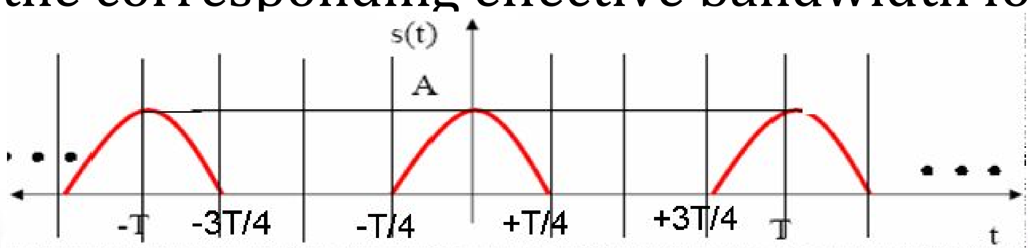
Complete Fourier Analysis Example

- Consider the half-wave rectified cosine signal, Figure 2.17:



$$\frac{A}{\pi}\left(1 + \frac{\pi}{2}\cos \omega_1 t + \frac{2}{3}\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_1 t + \frac{2}{35}\cos 6\omega_1 t \ldots (-1)^{\frac{n}{2}+1}\frac{2}{n^2-1}\cos n\omega_1 t + \ldots\right) \quad n \text{ even}$$

  - Write a mathematical expression for s(t) over its period T
  - Compute the Fourier series for s(t) (Amplitude & Phase form)
  - Get an expression for the power spectral density function for s(t)
  - Find the total power of s(t) from the time domain
  - Find the order of the highest harmonic *n* such that the Fourier series for s(t) contains at least 95% of the total signal power
  - Determine the corresponding effective bandwidth for the signal

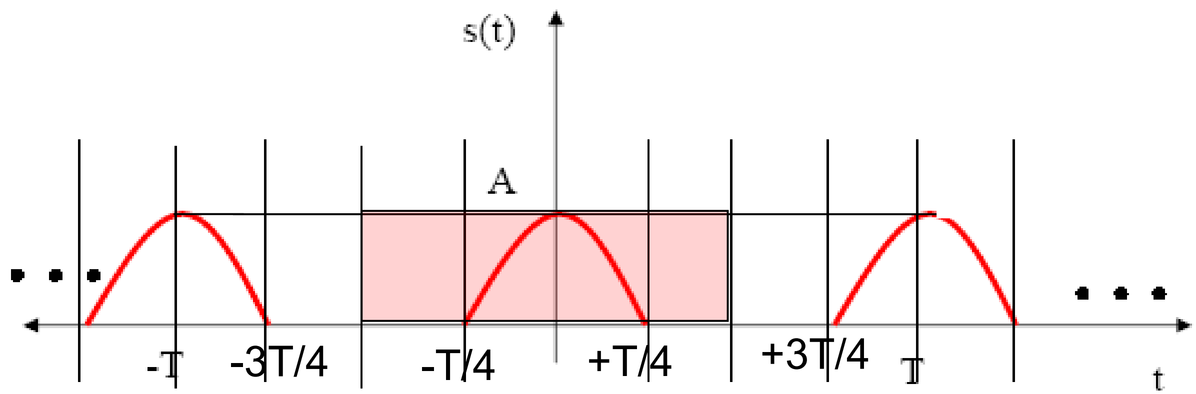


Example (Cont.)

1. Mathematical expression for s(t):

$$s(t) = \begin{cases} A\cos(2\pi f_o t) & , -T/4 \le t \le T/4 \\ 0 & , T/4 \le t \le 3T/4 \end{cases}$$

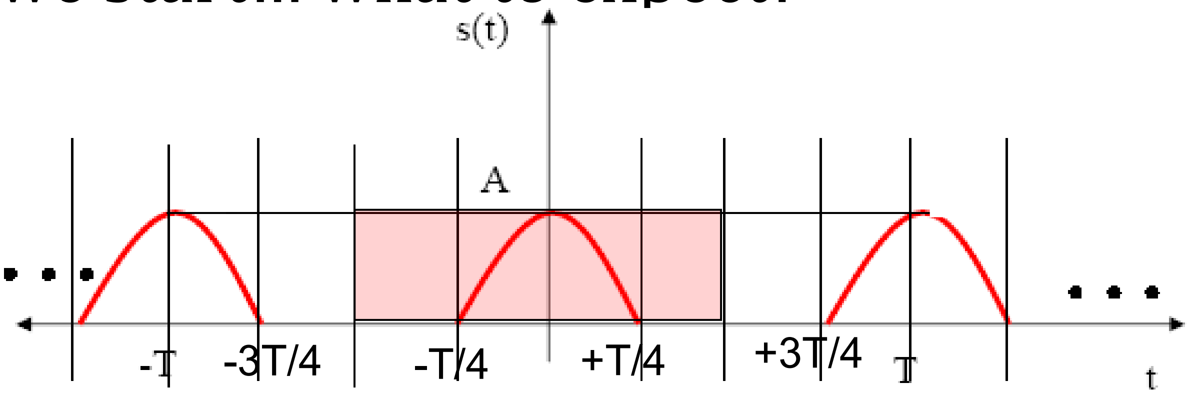
Where f_0 is the fundamental frequency, $f_0 = (1/T)$



Example (Cont.)

2. Fourier series

Before we start… what to expect?



- DC Component?
- Even or odd function?
- A_0 ?
- A_n ?
- B_n ?

Sine/cosine form of the Fourier Series

$$s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t) \right]$$

To get to the amplitude-phase form of the Fourier series, we must first obtain the sine-cosine form

Example (Cont.)

Fourier Analysis: $$A_0 = \frac{2}{T}\int_0^T x(t)dt$$

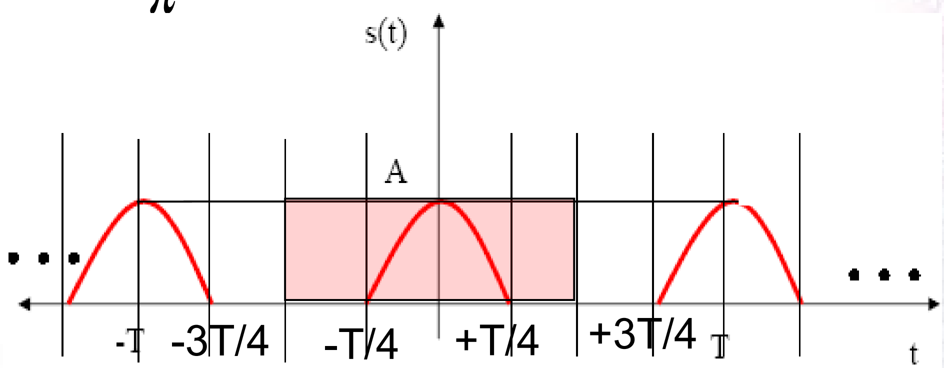
$f_0 = (1/T)$

$$A_0 = \frac{2}{T}\int_{-T/4}^{T/4} s(t)dt = \frac{2A}{T}\int_{-T/4}^{T/4}\cos(2\pi f_o t)dt$$

$$= \frac{2A}{T}\times\left[\frac{\sin(2\pi t/T)}{2\pi/T}\right]_{t=-T/4}^{t=T/4} = \frac{A}{\pi}\times\left[\sin(\pi/2)-\sin(-\pi/2)\right]$$

$$= \frac{A}{\pi}\times\left[\sin(\pi/2)+\sin(\pi/2)\right] = \frac{A}{\pi}\times\left[2\times\sin(\pi/2)\right]$$

$$= \frac{2A}{\pi} \quad \text{, as } \sin(\pi/2) = 1$$



DC = ?

Example (Cont.)

1. Fourier Analysis (cont.):

$$A_n = \frac{2}{T}\int_0^T x(t)\cos(2\pi n f_0 t)dt$$

$f_0 = (1/T)$

$$A_n = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\cos(2\pi n f_o t)dt = \frac{2A}{T}\int_{-T/4}^{T/4}\cos(2\pi f_o t)\cos(2\pi n f_o t)dt$$

$$= \frac{2A}{T}\times\left[\frac{\sin(2\pi(n+1)f_o t)}{4\pi(n+1)f_o} + \frac{\sin(2\pi(n-1)f_o t)}{4\pi(n-1)f_o}\right]_{-T/4}^{T/4}, \text{ for } n \neq 1$$

$$= \frac{A}{\pi}\times\left[\frac{\cos(n\pi/2)}{(n+1)} + \frac{-\cos(n\pi/2)}{(n-1)}\right], \text{ for } n \neq 1$$

n = 1 will be treated Separately later

Note: $\int \cos(ax)\cos(bx)dx = \frac{\sin(ax+bx)}{2(a+b)} + \frac{\sin(ax-bx)}{2(a-b)}$, and

$\sin(x) = \cos(x - \pi/2)$ From integral tables

Example (Cont.)

Fourier Analysis (cont.):

n ≠ 1

$$A_n = 0 \qquad , \text{ for } n \text{ odd and } n \neq 1$$

$$A_n = \frac{A}{\pi} \times \left[\frac{(-1)^{(n/2)}}{(n+1)} + \frac{-(-1)^{(n/2)}}{(n-1)} \right] \qquad , \text{ for } n \text{ even}$$

$$= \frac{A}{\pi} \times \left[\frac{(-1)^{(n/2)}(n-1) + (-1)(-1)^{(n/2)}(n+1)}{(n+1)(n-1)} \right]$$

$$= \frac{A(-1)^{(n/2)}}{\pi (n^2 - 1)} \times \left[(n-1) + (-1)(n+1) \right]$$

$$= \frac{2A(-1)^{(1+n/2)}}{\pi (n^2 - 1)} \qquad , \text{ for } n \text{ even}$$

Example (Cont.)

Fourier Analysis (cont.):

For n = 1, A_1 is obtained separately

$$A_{n=1} = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\cos(2\pi \times 1 \times f_o t)dt = \frac{2A}{T}\int_{-T/4}^{T/4} \cos(2\pi f_o t)\cos(2\pi f_o t)dt$$

$$= \frac{2A}{T}\int_{-T/4}^{T/4} \cos^2(2\pi f_o t)dt$$

$$= \frac{2A}{T} \times \left[\frac{t}{2} + \frac{\sin(4\pi f_o t)}{2 \times 4\pi f_o}\right]_{-T/4}^{T/4} = \frac{2A}{T} \times \left[\frac{T}{4} + \frac{\sin(\pi) - \sin(-\pi)}{8\pi f_o}\right]$$

$$= \frac{A}{2}$$

f0 = 1/T

Note: $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

Example (Cont.)

$$B_n = \frac{2}{T}\int_0^T x(t)\sin(2\pi n f_0 t)dt$$

Fourier Analysis (cont.):

$$B_n = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\sin(2\pi n f_o t)dt = \frac{2A}{T}\int_{-T/4}^{T/4} \cos(2\pi f_o t)\sin(2\pi n f_o t)dt$$

$$= \frac{-2A}{T} \times \left[\frac{\cos(2\pi (n+1) f_o t)}{4\pi (n+1) f_o} + \frac{\cos(2\pi (n-1) f_o t)}{4\pi (n-1) f_o} \right]_{-T/4}^{T/4}, \text{ for } n \neq 1$$

$$= 0 \qquad , \text{ for } n \neq 1$$

Note: $\int \cos(bx)\sin(ax)dx = \frac{-\cos(ax+bx)}{2(a+b)} - \frac{\cos(ax-bx)}{2(a-b)}$

Example (Cont.)

Fourier Analysis (cont.):

For n = 1, B_1 is obtained separately

$$B_{n=1} = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\sin(2\pi \times 1 \times f_o t)dt = \frac{2A}{T}\int_{-T/4}^{T/4} \cos(2\pi f_o t)\sin(2\pi f_o t)dt$$

$$= \frac{A}{T}\int_{-T/4}^{T/4} \sin(4\pi f_o t)dt$$

$$= \frac{-A}{4\pi} \times \left[\cos(4\pi f_o t)\right]_{-T/4}^{T/4} = \frac{-A}{4\pi} \times \left[\cos(\pi) - \cos(-\pi)\right] = \frac{-A}{4\pi} \times \left[\cos(\pi) - \cos(\pi)\right]$$

$$= 0$$

i.e. $B_n = 0$ for all n (our function is even!)

Example (Cont.)

Fourier Analysis (cont.):

$$s(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(2\pi n f_o t) + B_n \sin(2\pi n f_o t)\right]$$

$$= \frac{A}{\pi} + \frac{A}{2}\cos(2\pi f_o t) + \frac{2A}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{(-1)^{(1+n/2)}}{n^2-1}\cos(2\pi n f_o t)$$

$C_0 = A_0 \qquad C_n = \sqrt{A_n^2 + B_n^2} = A_n,$ since $B_n = 0$ *for all* n

$$C_o = \frac{2A}{\pi}, \quad C_1 = \frac{A}{2}$$

$$C_n = 0 \qquad , n \text{ is odd and } n \neq 1$$

$$C_n = \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)} \qquad , n = 2, 4, 6, \ldots$$

Note: θ_n are not required for PSD and power calculations

Example (Cont.)

$$C_o = \frac{2A}{\pi}, \quad C_1 = \frac{A}{2}$$

$$C_n = 0 \quad, n \text{ is odd and } n \neq 1$$

$$C_n = \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)} \quad, n = 2, 4, 6, \ldots$$

3. Power Spectral Density function (PSD):

$$x(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \left[C_n \cos(2\pi n f_0 t + \theta_n)\right]$$

$$PSD = \frac{C_0^2}{4} \times \delta(f) + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 \times \delta(f - nf_o)$$

$$= \frac{A^2}{\pi^2} \times \delta(f) + \frac{A^2}{8} \times \delta(f - f_o) + \frac{2A^2}{\pi^2} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\delta(f - nf_o)}{(n^2-1)^2}$$

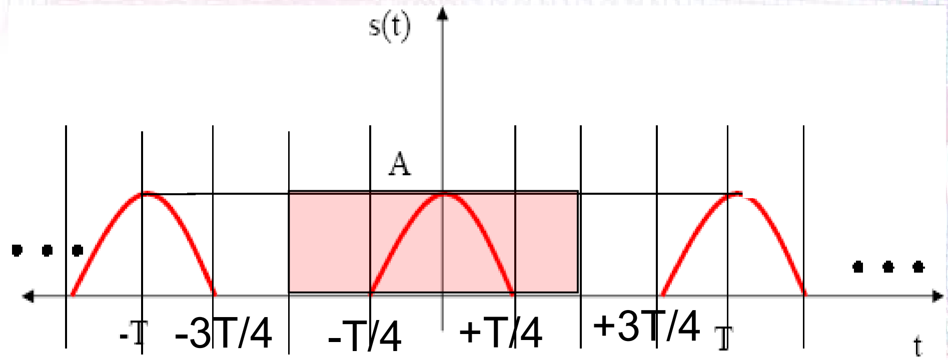
n = 0 (DC) n = 1 n = Even

For large n, power decays ∝ $(1/n^4)$… Good or bad?

Example (Cont.)

4. Total Power:

(From the time domain)



$$P_s = \frac{1}{T}\int_{-T/4}^{3T/4} |s(t)|^2\, dt = \frac{A^2}{T} \times \int_{-T/4}^{T/4} \cos^2(2\pi\ f_o t)dt$$

Note: $\cos^2\theta = ½(1 + \cos 2\theta)$

$$= \frac{A^2}{T} \times \left[\frac{t}{2} + \frac{\sin(4\pi\ f_o t)}{8\pi\ f_o t}\right]_{-T/4}^{T/4}$$

Zero

$$= \frac{A^2}{4} \quad = 0.25\, A^2$$

= ½ ($A^2/2$) = Half the power of a full sine wave

Example (Cont.)

5. Finding n such that we get at least 95% of the total power:

For

$$\Rightarrow \; PSD_{n=0} = \frac{C_0^2}{4} = \frac{4A^2}{4\pi^2} = \frac{A^2}{\pi^2} = 0.1014A^2$$

$$\Rightarrow \; Power\% = \frac{0.1014A^2}{0.25A^2} = 40.5\%$$

% of total power in this component

Example (Cont.)

Finding *n* such that we get at least 95% of the total power, contd.:

For

$$\Rightarrow \; PSD_{n=1} = \frac{C_0^2}{4} + \frac{C_1^2}{2} = \frac{A^2}{\pi^2} + \frac{A^2}{8} = 0.226A^2$$

$$\Rightarrow \; Power\% = \frac{0.226A^2}{0.25A^2} = 90.5\%$$

% of total power in these two components

Example (Cont.)

Finding *n* such that we get at least 95% of the total power, Contd.:

For

$$\Rightarrow \ PSD_{n=2} = \frac{C_0^2}{4} + \frac{C_1^2}{2} + \frac{C_2^2}{2} = \frac{A^2}{\pi^2} + \frac{A^2}{8} + \frac{2A^2}{9\pi^2} = 0.2485A^2$$

$$\Rightarrow \ Power\% = \frac{0.2485A^2}{0.25A^2} = 99.41\%$$ OK! ≥ 95%

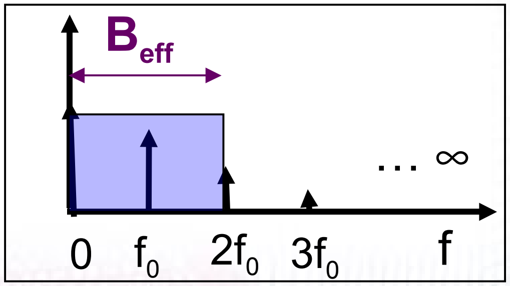
∴ n = 2, and

6. the effective bandwidth is:

$B_{eff} = f_{max} - f_{min}$

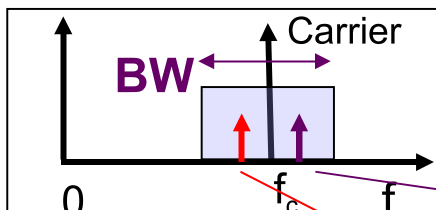
$B_{eff} = 2f_0 - 0 = 2f_0$

DC



Bandwidth about a Centre Frequency

- So far we have considered signals in their baseband form (without modulation)
- Data is often sent as variations in a high frequency carrier signal having a frequency f_c (modulation)
- So, bandwidth (BW) of this signal occupies a range of frequencies **centred** around f_c



With Amplitude Modulation,
For each component of the modulating signal:

Carrier

Modulating Signal

$$2\cos(2\pi f_c t)\cos(2\pi f_m t) = \cos[2\pi (f_c - f_m)t] + \cos[2\pi (f_c + f_m)t]$$

- The larger f_c, the larger the BW obtainable
- Largest BW obtainable for a given centre frequency f_c is 2 f_c